COMPARATIVE STUDY OF THE X−11 AND BAYSEA PROCEDURES OF SEASONAL ADJUSTMENT

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Abstract

A Bayesian seasonal adjustment procedure BAYSEA leads to a regression-type procedure which allows effective use of prior information of each particular time series. The decision on the selection of the basic model is realized by minimizing an objectively defined criterion. Numerical results show the relative advantage of the BAYSEA procedure over the X−11 procedure.

KEY WORDS

Seasonal adjustment	X−11
BAYSEA	Bayesian model
Trend	Seasonal

1. INTRODUCTION

The Census method of seasonal adjustment, particularly the X−11 variant of the method II, made a significant contribution to the practice of economic data analysis. The method is a typical example of how an appropriate accumulation of human experiences can lead to the development of an extremely useful data-analytic procedure. The construction of the procedure has often been questioned because of its lack of the basic statistical model. Nevertheless, the widely spread use of the X−11 constitutes a proof that it is a good practical procedure for the seasonal adjustment of economic data.

Since the social impact of the seasonal adjustment of an economic time series is often quite significant, it is natural that the improvement of the technique has been continuously contemplated. A useful reference on this subject is the paper by Kallek (1978) who discusses the general objectives and necessary improvements of the technique of seasonal adjustment. Also useful is the paper by Shiskin and Plewes (1978) who discuss the seasonal adjustment of the U.S. unemployment rate. These two papers are very informative as they explain the problems of seasonal adjustment based on the experiences of the authors. Of particular interest is the candid description of the purpose of seasonal adjustment by Kallek (1978, p. 15) who states that "one attempts to remove as much of the fluctuation which obscures the trend-cycle component of the series." We also notice that the important common observations of these authors are that a procedure must be equipped with various options which allow the incorporation of prior information related to a time series and that decisions on the selection of particular options must be based on objectively defined criteria.

In 1980, Hirotugu Akaike (1980a) proposed the use of a Bayesian model for seasonal adjustment and demonstrated its feasibility by numerical examples. The necessary computational procedure was then published in a computer program called BAYSEA (a Bayesian Seasonal Adjustment Program; Akaike and Ishiguro 1980a). In contrast to the Census method which is based on the moving average technique, this approach leads to a regression technique with a very simple structure and easily allows the incorporation of various options in the form of an increased number of independent variables. Also, since it is based on an explicitly defined statistical model, a very useful objective criterion for the selection of the best model is available. Thus, the procedure satisfies the two important requirements mentioned above, the availability of options and the objectivity of decision criteria, and it is only whether it really produces good adjustments of economic time series that determines the viability of this new procedure. Since there is no absolute criterion yet available for the evaluation of a seasonal adjustment procedure, the most natural starting point for the evaluation of this procedure will be the comparison of its performance with that of the best

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established, the X−11 procedure. This comparison is the subject of this paper.

The X−11 ARIMA developed by Dagum (1978) differs from the original X−11 as it extrapolates the original series by the ARIMA modeling. However, since the difference between the outputs of these two procedures is usually negligibly small compared with the difference between those of the original X−11 and BAYSEA, we will restrict our attention to the comparison with the original X−11. Only the statistics of the amount of revisions by the X−11 ARIMA when additional years of observations are added will be shown in section 5.

2. BASIC STRUCTURE OF THE BAYSEA PROCEDURE

As the basic model of the time series for seasonal adjustment we assume the additive structure

$$y(i) = T_i + S_i + I_i$$

where $y(i)$ denotes the observation at time i and T_i, S_i and I_i denote the trend, seasonal and irregular component, respectively. For simplicity of explanation, the series is considered to be of monthly observations.

What we typically expect of the two systematic components are that, at least locally, the trend component T_i will show a smooth behavior and that the yearly pattern of the seasonal component S_i will be stable. This expectation is formally represented by asking

$$[s^2(T_i - 2T_{i-1} + T_{i-2})^2 + (S_i - S_{i-12})^2 + z^2(S_i + S_{i-1} + \dots + S_{i-11})^2]$$

to be small. The last term within the brackets is added to keep the 12-month sum of the seasonal variations close to zero. Here s and z are properly chosen constants. We also expect that the systematic part $T_i + S_i$ will not deviate significantly from the original observation $y(i)$. This suggests the minimization of

$$[y(i) - T_i - S_i]^2$$

The simultaneous minimization of the above two quantities leads to the minimization of

$$\{[y(i) - T_i - S_i]^2 + d^2\,[s^2(T_i - 2T_{i-1} + T_{i-2})^2 + (S_i - S_{i-12})^2 + z^2(S_i + S_{i-1} + \dots + S_{i-11})^2]\}$$

where d is a properly chosen constant.

If we denote by y the vector of the observations $(y(N), y(N-1), \dots, y(1))'$ and by c the vector of the seasonal and trend components $(S_N, S_{N-1}, \dots, S_1, T_N, T_{N-1}, \dots, T_1)'$ where (′) denotes transpose, the above quantity can be represented by

$$L(c) = \|y - Xc\|^2 + \|D(c - c_0)\|^2$$

where $\|x\|$ denotes the Euclidean norm $(x_1^2 + x_2^2 + \dots + x_k^2)^{1/2}$ of a vector $x = (x_1, x_2, \dots, x_k)$. The matrices X and D are given by

$$X = [I_N \; I_N] \text{ and}$$

$$D = d \cdot \begin{bmatrix} D_{11} & 0 \\ zD_{12} & 0 \\ 0 & sD_{32} \end{bmatrix}$$

where I_N denotes an $N \times N$ identity matrix and

$$D_{11} = \begin{array}{c|cccccccccc|} & 1 & & & & 13 & & & & N \\ & 1 & 0 & & & -1 & & & & \\ & & 1 & 0 & & & -1 & & 0 & \\ & & & \cdot & 0 & & & \cdot & & \\ & & & & \cdot & & & & \cdot & \\ & & & & & \cdot & & & & \cdot \\ & & & & & & 1 & 0 & & -1 \\ & & & & & & & \cdot & & \cdot \\ & & 0 & & & & & & \cdot & \cdot \\ & & & & & & & & 1 & 0 \\ & & & & & & & & & 1 \end{array}$$

$$D_{12} = \begin{array}{c|cccccccccc|} & 1 & & & & 12 & & & & N \\ & 1 & 1 & \cdot & \cdot & 1 & & & & \\ & & 1 & 1 & \cdot & \cdot & 1 & & & \\ & & & \cdot & & & & \cdot & & \\ & & & & \cdot & & & & \cdot & \\ & & & & & \cdot & & & & \cdot \\ & & & & & & 1 & 1 & \cdot \; \cdot & 1 \\ & & & & & & & \cdot & & \cdot \\ & & 0 & & & & & & \cdot & \cdot \\ & & & & & & & & 1 & 1 \\ & & & & & & & & & 1 \end{array}$$

$$D_{32} = \begin{array}{c|cccccccc|} & 1 & & & & & & & N \\ & 1 & -2 & 1 & & & & & \\ & & 1 & -2 & 1 & & 0 & & \\ & & & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & \cdot & & \\ & & & & & \cdot & \cdot & \cdot & \\ & & 0 & & & & 1 & -2 & 1 \\ & & & & & & & 1 & -2 \\ & & & & & & & & 1 \end{array}$$

Obviously, we may consider the use of higher or lower order differences of the trend and seasonal components than those used in the above definition. The necessary modification of D is straightforward. The vector c_0 represents the effect of the past trend and seasonal components T_0, T_{-1}, T_{-2} and S_0, $S_{-1}, \dots, S_{-11}$. The best choice c_* of c is then defined as that which minimizes the sum of squares $L(c)$.

Since we have $L(c) = \|y_* - X_*c\|^2$, where

$$y_* = \begin{bmatrix} y \\ Dc_0 \end{bmatrix} \qquad X_* = \begin{bmatrix} X \\ D \end{bmatrix}$$

the solution c_* to the above problem can easily be obtained by any standard procedure of least squares. This completes the description of the basic structure of BAYSEA. One remarkable characteristic of the procedure is that it is not bothered by the problem of extrapolation at the end of the time series.

3. THE BAYESIAN MODELING AND THE SELECTION CRITERION

Although the basic structure is simple, BAYSEA contains several parameters which must be specified before it becomes operational. We solve this problem as a problem of statistical model selection. For this purpose, first we note the simple fact that the minimization of $L(c)$ is identical to the maximization of

$$\exp[-L(c)/2v)] = \exp[-\|y - Xc\|^2/(2v)]$$
$$\exp[-\|c - c_0\|^2/(2v)]$$

where v is a positive constant. This shows that the solution c_* of the least squares problem is the mode or the mean of the posterior destribution of the Bayesian model defined with the data distribution

$$f(y|v,c) = 2\pi v)^{-N/2} \exp[-\|y - Xc\|^2/(2v)]$$

and the prior distribution

$$p(c|v,D,c_0) = (2\pi v)^{-N/2}|D'D|^{1/2}$$
$$\exp[-\|D(c - c_0)\|^2/(2v)]$$

The relative goodness of a Bayesian model can be measured by the likelihood of the model defined by

$$M(v,D,c_0) = \int f(y|v,c)p(c|v,D,c_0)\,dc$$

By a simple calculation we get

$$M(v,D,c_0) = (2\pi v)^{-N/2}|R|^{1/2}|X'X + R|^{-1/2}$$
$$\exp[-L(c_*)/(2v)]$$

where $R = D'D$. As the estimate of the variance of the irregular component I_i, we adopt the value of v that maximizes the likelihood. This is given by $v_0 = (1/N)\,L(c_*)$. For comparison of models defined with different D's, we adopt

$$\text{ABIC} = (-2)\ ln\ M(v_0,D,c_0)$$
$$= N\ ln\ v_0 + ln\ |X'X + R| - ln\ |R| + N\ \text{const.}$$

where ln denotes natual logarithm. A model with a smaller value of ABIC is considered to be a better model. Here ABIC stands for a Bayesian information criterion and is so called due to its similarity to the criterion AIC developed for the comparison of statistical models with parameters determined by the method of maximum likelihood (Akaike 1980a).

4. IMPLEMENTATION OF THE BAYSEA PROCEDURE

In the present version of BAYSEA, the optimum choice of the scaling factor d of the matrix D is realized by a discrete search over the interval (1.0, 20.0) for the minimum of ABIC. The lower bound 1.0 was chosen from the consideration that for d less than 1.0 much of the irregular movement of the series will be absorbed into the trend and seasonal components. The choice of the upper bound 20.0 is based on the consideration that the signal to noise ratio 1/20, in terms of the amplitude ratio, is a limit for visual detection of the signal when it is buried in the noise. The parameter z which controls the average of the seasonal component is determined by the relation

$$z = p^{-1/2}s^{-1}$$

where p denotes the length of the fundamental period, which is equal to 12 for monthly data, s is the scaling factor of the matrix D_{32} which controls the smoothness of the trend component. Thus, the basic BAYSEA procedure is defined by the following three parameters:

ORDER = order of differencing of T_i
SORDER = order of differencing of S_i
RIGID = $1/s$

The naming of RIGID is due to the fact that the increase of the value $1/s$ increases the rigidity of the seasonal pattern.

By the computer program BAYSEA, the basic procedure is applied successively to blocks of data of length SPAN × PERIOD, where PERIOD denotes the value of p above defined. SPAN is set equal to 4 in our study. This choice is based on the observation that in the papers hitherto published, it is often mentioned that at the time when observations of additional 3 years are added, the result of the seasonal adjustment must be considered to be final. The prior mean c_0 is obtained as the mean of the vector c conditional on the past values of the trend and seasonal component. At the beginning of the computation, the basic procedure is applied to the first block of data of length (2 × SPAN − 1) × PERIOD and the necessary past trend and seasonal components are determined by an approximate method of maximum likelihood. In the actual implementation of BAYSEA, the estimates of the trend and seasonal components before the present block of data are fixed and treated as the true values to define c_0. For the details of the computation of c_0, readers are referred to Akaike (1980b). The block of data for analysis is successively shifted by the number of data points equal to SHIFT × PERIOD. In the present study, SHIFT is set equal to 1.

5. NUMERICAL RESULTS

The search for the optimal model by BAYSEA was limited to the following five models defined by the combinations of the parameters; (ORDER, SORDER, RIGID) = (1, 1, 1.0), (2, 1, 1.0), (2, 1, 0.5), (2, 2, 1.0) and (2, 2, 0.25). The procedures X−11 and BAYSEA were applied to the 13 sets of economic data provided by the Bureau of the Census. Both procedures have options for the correction of extreme values or outliers. However, since these options will be effective only when the basic procedures work satisfactorily to give good initial estimates of the irregular component, we limited our attention mainly to the comparison of the procedures without these options. The additive version of X−11 was applied to tables 1-3 and the multiplicative version to the tables 4-13. Correspondingly, the BAYSEA was applied to tables 4-14 after natural log transformation. The numerical results obtained by the X−11 for tables 4-14 were all transformed into natural logarithms for comparison. The choice of additive or log additive model by the BAYSEA can be based on ABIC with the necessary modification of the likelihood for the log transformation.

Throughout the rest of this paper the following abbreviated notations are used:

X−11 OTL:	X−11 with standard outlier correction
X−11 NOTL:	X−11 without outlier correction
X−11 ARIMA NOTL:	X−11 ARIMA with 1 year of forecasts and without outlier correction
BAYSE(*m*,*n*,*r*):	BAYSEA with ORDER=*m*, SORDER=*n* and RIGID=*r*
ST_i, SS_i, SI_i:	Sum of squares of the amount of revisions of the trend, seasonal and irregular component, respectively, due to the inclusion of *i* additional years of observations.

The computation was performed by starting with the first 7-years' data as the initial set and then adding the set of the following complete 1-year data successively.

In the following, we will describe the numerical results.[1] Graphical results were obtained by the X−11 and BAYSEA without outlier correction options and they are included at the end of the paper. The Bayesian models adopted for BAYSEA were selected by minimizing the criterion AVABIC, the average of ABIC of each data block over the total span of data.

[1] To produce the real values, numbers in each table should be multiplied by the number within the parentheses located directly below the table.

Table 1. AGRICULTURAL EMPLOYMENT, MEN, 20 YEARS OLD AND OLDER (BLSAGEMEN)

	ST1	ST3	SS1	SS3	SI1	SI3
X−11 OTL	0.84	1.47	1.19	2.33	1.10	0.85
X−11 NOTL	.72	1.24	1.18	2.08	.78	.79
X−11 ARIMA NOTL	.63	.97	.90	1.43	.40	.53
BAYSEA (1, 1, 1.0)	.59	.75	.80	1.00	.20	.19

($x\ 10^4$)

Comments: Figure 1 shows that the results obtained by X−11 NOTL and the BAYSEA are very close. The behavior of the trend by the BAYSEA looks slightly more irregular than that by X−11 NOTL. The revisions are smaller for the BAYSEA than for the X−11 procedures. This is a typical result by BAYSEA (1, 1, 1.0).

Table 2. UNEMPLOYED WOMEN, 16 TO 19 YEARS OLD (CPS DATA, BLSUEW 16-19)

	ST1	ST3	SS1	SS3	SI1	SI3
X− OTL	0.17	0.32	0.47	1.01	0.54	1.32
X−11 NOTL	.36	.34	.65	.99	.74	1.05
X−11 ARIMA NOTL	.45	.44	.68	.96	.75	1.04
BAYSEA (2, 2, 0.25)	.64	.75	1.06	1.03	1.22	1.48

($x10^4$)

Comments: Figure 2 shows that apparently the final estimates of the seasonal component by both procedures are very close. This makes the smaller revision of the trend by the X−11, represented by the lower values of ST1 and ST3, rather attractive. However, the trend of X−11 NOTL is very wavy and suggests the necessity of further analysis. This will become the main point of the present comparative study.

Table 3. ALL EMPLOYEES IN FOOD INDUSTRIES (ESTABLISHMENT DATA, BLSALLFOOD)

	ST1	ST3	SS1	SS3	SI1	SI3
X−11 OTL	0.85	0.94	0.32	0.85	0.49	0.71
X−11 NOTL	.76	1.03	.50	1.03	.36	.47
X−11 ARIMA NOTL	1.06	1.54	.80	1.14	.39	.66
BAYSEA (2, 1, 1.0)	.25	.44	.21	.29	.36	.56

($x10^3$)

Comments: Figure 3 shows that apparently the BAYSEA is producing a result similar to that by X−11 NOTL. However, BAYSEA is producing a more stable seasonal and the low values of ST3 and SS3 make the procedure very attractive. This is a typical result by BAYSEA (2, 2, 1.0).

Table 4. DEPOSIT COMPONENT OF M−1A MONEY SUPPLY (DEMANDEPOSIT)

	ST1	ST3	SS1	SS3	SI1	SI3
X− OTL	1.20	1.06	0.99	2.42	1.04	1.70
X−11 NOTL	.83	.95	.93	1.70	.30	.71
X−11 ARIMA NOTL	.86	.93	1.08	1.58	.39	.68
BAYSEA (2, 2, 1.0)	.82	.74	.99	1.28	1.09	1.02

(x10^{-4})

Comments: The magnitudes of the revisions by the BAYSEA are similar to those by the X−11 procedures. This is a typical result by BAYSEA (2, 2, 1,0). The parameter SORDER = 2 was required to follow the moving seasonality. As can be seen from figure 4, the results by the two procedures are very close. Although invisible in the figure, the trend of the BAYSEA is smoother than that of X−11 NOTL. Note that the results of figures 4-9 are given in terms of natural logarithms.

Table 5. CURRENCY COMPONENT OF M−1A MONEY SUPPLY (CURRENCYM1A)

	ST1	ST3	SS1	SS3	SI1	SI3
X−11 OTL	0.16	0.16	0.20	0.29	0.18	0.31
X−11 NOTL	.41	.33	.21	.16	.57	.63
X−11 ARIMA NOTL	.38	.29	.18	.17	.57	.63
BAYSEA (2, 1, 1,0)	.11	.35	.03	.06	.16	.53

(x10^{-4})

Comments: The data have a very smooth trend and constant seasonality. The revision of the seasonal component by the BAYSEA is very small. This is a characteristic of BAYSEA (2, 1, 1.0). The results by X−11 NOTL and the BAYSEA are so close that ther is no point in reproducing them in the form of a figure. This makes the low vallues of the revisions of the BAYSEA rather attractive. There was some indication of a trading-day effect.

Table 6. RETAIL SALES OF WOMEN'S APPAREL (RSWOMEN)

	ST1	ST3	SS1	SS3	SI1	SI3
X−11 OTL	0.23	0.53	0.52	1.07	0.63	0.87
X−11 NOTL	.15	.18	.45	.90	.39	.59
X−11 ARIMA NOTL	.18	.19	.52	.87	.45	.66
BAYSEA (2, 2, 0.25)	.64	.69	.47	.96	.76	1.24

(x10^{-2})

Comments: The revisions of the BAYSEA estimates are comparable to or larger than those of the X−11 estimates. This is a typical result by BAYSEA (2, 2, 0.25). The trend of the BAYSEA is much smoother than that of X−11 NOTL and the comments on table 2 also apply to this example.

Table 7. WHOLESALE INVENTORIES OF GROCERY STORES (WIGROCERY)

	ST1	ST3	SS1	SS3	SI1	SI3
X−11 OTL	0.31	0.31	0.15	0.24	0.21	0.23
X−11 NOTL	.32	.29	.19	.34	.25	.31
X−11 ARIMA NOTL	.40	.39	.29	.40	.22	.32
BAYSEA (2, 1, 1.0)	.15	.14	.06	.07	.15	.19

(x10^{-2})

Comments: The revisions of the BAYSEA estimates are very small. As we mentioned in our comments on table 5 this is a characteristic of the model BAYSEA (2, 1, 1.0). The results of X−11 NOTL and the BAYSEA are close.

Table 8. RETAIL SALES OF AUTOMOTIVE DEALERS (RAUTODLRS)

	ST1	ST3	SS1	SS3	SI1	SI3
X−11 OTL	1.80	2.67	1.15	2.37	1.72	3.74
X−11 NOTL	.91	1.13	1.25	2.18	.88	1.37
X−11 ARIMA NOTL	.64	.99	1.14	2.13	.75	1.26
BAYSEA (2, 2, 1.0)	1.04	1.58	1.01	.92	1.77	2.70

(x10^{-2})

Comments: As we noted in our comments on tables 2, 4, and 6, the relatively large revisions of the trend and irregular of the BAYSEA are due to the choice ORDER = SORDER = 2. Figure 5 shows that the BAYSEA is producing a much smoother trend and more stable seasonal than X−11 NOTL. There was some indication of a trading-day effect.

Table 9. VALUES OF SHIPMENTS, BLAST FURNACE AND STEEL MILLS (INS11VS)

	ST1	ST3	SS1	SS3	SI1	SI3
X−11 OTL	0.80	1.03	0.45	0.83	0.47	0.62
X−11 NOTL	.71	.97	.85	1.31	.58	.70
X−11 ARIMA NOTL	.70	.96	.85	1.31	.58	.70
BAYSEA (2, 1, 0.5)	1.19	2.05	.64	.86	1.62	2.78

(x10^{-1})

Comments: This is a series with typical irregularities due to strikes. Figure 6 shows that the trend of the BAYSEA is rather insensitive to the strikes in the years 1959, 1962, 1968, and 1971. The seasonal component of the BAYSEA is more stable than that of X−11 NOTL. The irregular component of the BAYSEA is more stable than that of X−11 NOTL. THe irregular component of the BAYSEA shows the effect of strikes more clearly than that of X−11 NOTL. This is an example which shows that small values of revisions of the trend and irregular do not necessarily mean that the procedure is performing properly.

Table 10. VALUE OF UNFILLED ORDERS, RADIO AND TV (INS36U0)

	ST1	ST3	SS1	SS3	SI1	SI3
X−11 OTL	2.83	3.16	0.72	3.24	0.99	2.88
X−11 NOTL	.42	.80	.69	1.58	.48	.80
X−11 ARIMA NOTL	.47	.79	.69	1.50	.52	.84
BAYSEA (2, 2, 1.0)	.57	.79	.75	.85	1.08	1.35

($x10^{-1}$)

Comments: The relatively large revisions of the BAYSEA estimates are due to thc choice ORDER = SORDER = 2. The general impression of the difference between the results by X−11 NOTL and the BAYSEA is similar to that given by figure 5 of table 8.

Table 11. VALUE OF SHIPMENTS, BEVERAGES (INS62VS)

	ST1	ST3	SS1	SS3	SI1	SI3
X−11 OTL	1.27	.188	1.24	2.66	1.68	2.61
X−11 NOTL	1.44	1.87	1.57	2.77	1.47	1.98
X−11 ARIMA NOTL	1.54	1.90	1.58	2.59	1.45	1.83
BAYSEA (2, 2, 0.25)	2.01	1.96	2.54	2.47	3.46	3.31

($x10^{-1}$)

Comments: This is another example which typically shows the relatively large revisions of estimates by BAYSEA (2, 2, 0.25). However, figure 7 shows that the trend produced by X−11 NOTL displays very erratic fluctuations. The power spectra given in figure 8 whos that by use of X−11 NOTL, too much of the power of the irregular component around the frequency one cycle/year is eliminated and allocated to the trend component. The spectra were obtained by fitting 30th order AR model to the data.

Table 12. HOUSING STARTS, SOUTH, SINGLE-FAMILY DWELLINGS (CON-HSS1F)

	ST1	ST3	SS1	SS3	SI1	SI3
X−11 OTL	0.19	0.44	0.50	0.88	0.42	0.63
X−11 NOTL	.25	.48	.52	.99	.42	.61
X−11 ARIMA NOTL	.28	.51	.51	.97	.41	.59
BAYSEA (2, 2, 1.0)	.29	.41	.28	.34	.42	.60

($x10^{-1}$)

Comments: The general impression of the differences of the results by X−11 NOTL and the BAYSEA is quite similar to that given by figure 5 of table 8. Thus, the smaller revision of the seasonal by the BAYSEA makes the procedure attractive.

Table 13. HOUSING STARTS, NORTH-CENTRAL, DWELLINGS OF FIVE OR MORE FAMILIES (CON-HSNCS)

	ST1	ST3	SS1	SS3	SI1	SI3
X−11 OTL	0.17	0.35	0.56	1.29	0.64	1.44
X−11 NOTl	.25	.27	.46	.76	.52	.72
X−11 ARIMA NOTL	.25	.27	.46	.76	.52	.73
BAYSEA (2, 2, 0.25)	.39	.44	.49	.56	.79	.92

($x10^{0}$)

Comments: The choice of the parameters of BAYSEA is due to the moving seasonality towards the end of the data. (See fig. 9.) The trend of the BAYSEA is much smoother than that of X−11 NOTL. However, the trend of the BAYSEA looks too insensitive to the dip of the level around the years 1966 and 1967. This means that the present choice of RIGID = 0.25 is too low for this part of the data. This problem of structural change of the original time series will be discussed in more detail in the next section.

6. DISCUSSION

The numerical results given in the preceding section revealed remarkable similarity of the performance of the X−11 and BAYSEA procedures. However, significant differences were often observed between the trend components obtained by the two procedures. Our general impression was that X−11 was generating spurious fluctuations of trend. That this is the case is demonstrated by the following example of the time series of unemployment of the United States.

Figure 10 shows the results of applying the additive version of X−11 with the standard option for outlier correction and BAYSEA (2, 1, 1.0) with some ad hoc option to be described shortly to the time series of the number of the U.S. unemployed, 16 years of age and older, for the years 1972-78. In this example, we see the typical wavy movement observed in the preceding section in the trend produced by X−11. There is a sharp rise of the level at January 1975. The effect of this abrupt change on the seasonal adjustment is discussed in great detail by Shiskin and Plewes (1978). The use of some ad hoc preprocessing to compensate for the change of the level is suggested by these authors. By the present Bayesian approach, the abrupt change suggests an increase of the uncertainty of the variance of the corresponding prior distribution. The BAYSEA procedure, which produced the result illustrated in figure 10, was obtained by multiplying the five rows of the matrix D, which generated the second-order differences $T_i - 2T_{i-1} + T_{i-2}$ with i centered at January 1975, by the factor 0.05. The position of the central row and the value of the multiplicative factor were chosen by minimizing ABIC. The BAYSEA trend thus obtained shows a very smooth decline during 1977 that is in sharp contrast to the wavy behavior of the X−11 trend.

By comparing the seasonal components obtained by the two procedures, we can see that the particular behavior of the X−11 trend is related to the large amplitude of the X−11 seasonal. In the first half of each year, the positive swing of the X−11 seasonal corresponds to the downward swing of the trend, and in the second half, the negative swing of the seasonal corresponds to the upward swing of the trend. This observation shows that a spurious fluctuation of frequency around one cycle/year may be generated by the overadjustment of seasonality. Thus, we may conclude that the wavy pattern often observed in the trend components obtained by the X−11 procedure is quite probably an indication of the overadjustment.

Detailed study of this aspect will require a cross-spectrum analysis between the fluctuations of the trend and the seasonal component. Here, we will only note that the cross-correlations between the monthly differences of the trend and the yearly differences of the seasonal were often more significant, in the examples treated in the preceding section, for the X−11 than for the BAYSEA procedure. This observation of the necessity of looking at the trend component directly rather than the adjusted series, which is obtained by subtracting the seasonal component from the original series, is an important result of our present comparative study.

The X−11 and BAYSEA procedures were applied to the same series of the U.S. unemployment for the years 1967-1978 and the sums of squares of the revisions of the estimates were obtained as follows:

	ST1	ST3	SS1	SS3	SI1	SS3
X−11 OTL	6.00	5.35	2.17	4.99	4.93	4.45
X−11 NOTL	6.47	7.83	2.72	4.85	3.29	3.38
BAYSEA (2, 1, 1.0)	1.89	2.02	1.21	1.65	0.60	0.65
BAYSEA (2, 1, 1.0)[1]	1.14	1.32	0.60	0.59	1.05	1.11

$(\times 10^0)$

[1]With increased prior variances for the trend differences around January 1975.

We can see the drastic reduction of the amount of revisions of the trend and seasonal components by the introduction of the modification of BAYSEA. The very small values of the sums of squared revisions and the apparently very natural behavior of the trend illustrated in figure 10 strongly indicate the advantage of the modified BAYSEA over the standard X−11 procedure. The simplicity of the necessary operation and the clarity of the underlying motive of the present modification provide a good demonstration of the versatility of the BAYSEA procedure.

The behavior of the trend obtained by the BAYSEA for table 13 and illustrated in figure 9 might have given some impression that the procedure is too insensitive to the dip of the level of the original data in the years 1966-67. By inspection, it turned out that BAYSEA (2, 2, 1.0) was producing lower values of ABIC at this part of the data. This suggests that the time series changed its character from constant to moving seasonality during the period of observation. The BAYSEA procedure allows the detection of such a change of character through the comparison of ABIC's of various models. By simultaneously running models with different parameters and watching the behavior of ABIC's, we can easily identify the present status of the time series.

What is more important is the fact that exp(−0.5ABIC) plays the role of the likelihood of a model. Thus, in the case of table 13, by taking the average of the results of BAYSEA(2, 2, 0.25) and BAYSEA (2, 2, 1.0) for each block of data with weights proportional to the exp(−0.5ABIC)'s, we obtained the result shown in figure 11. The trend is now responding to the dip of the level at 1966-67, yet showing the adaptation to the moving seasonality towards the end of the data. This result suggests that by simultaneously running the five models used in this paper and taking the average of the results with weight proportional to the exp(−0.5ABIC)'s, one can get a very practical procedure of adaptive seasonal adjustment.

7. CONCLUSION

Options necessary for handling missing observations, trading-day effects, and other additive inputs are already developed for BAYSEA (Akaike 1979, Akaike and Ishiguro 1980b, and Ishiguro and Akaike 1980). Decisions on these options are also based on the objective criterion ABIC. The numerical results reported in this paper confirm that the BAYSEA procedure equipped with a set of a fairly small number of alternative models can produce results that are apparently often better than those of the X−11 procedure. The simplicity of the structure, the versatility of options, and the availability of an objective criterion for making decisions make the BAYSEA procedure quite an attractive alternative to the X−11 procedure.

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Figure 1. AGRICULTURAL EMPLOYMENT, MEN 20 YEARS OLD AND OVER

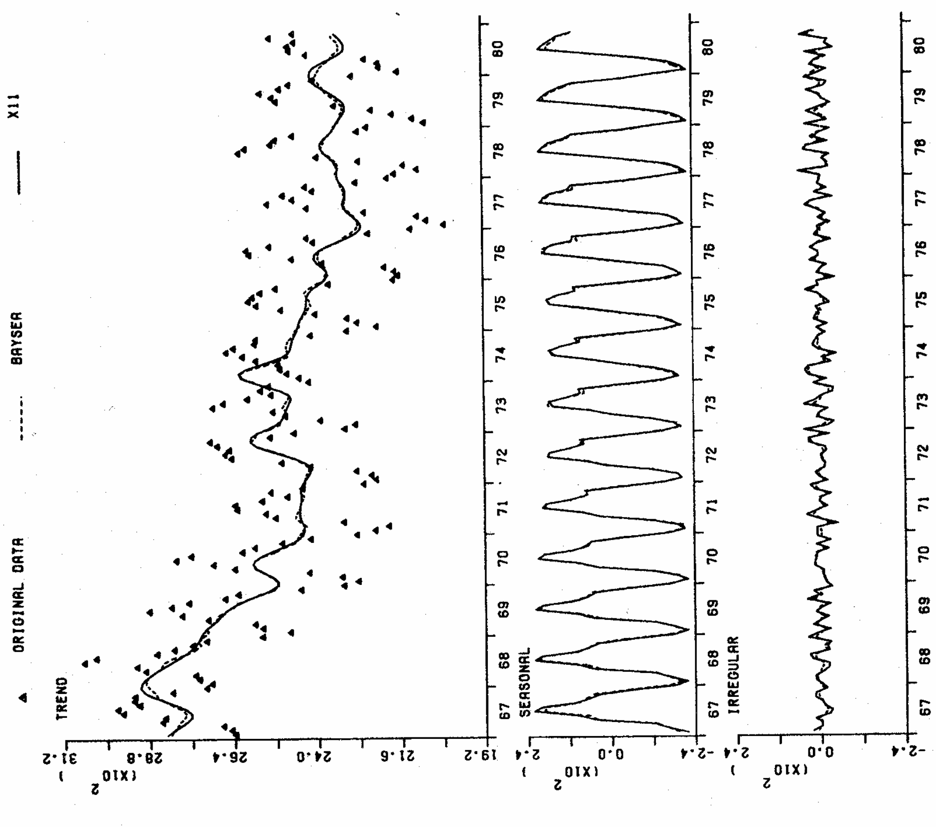


Figure 2. UNEMPLOYED WOMEN, 16-19 YEARS OLD
(CPS data)

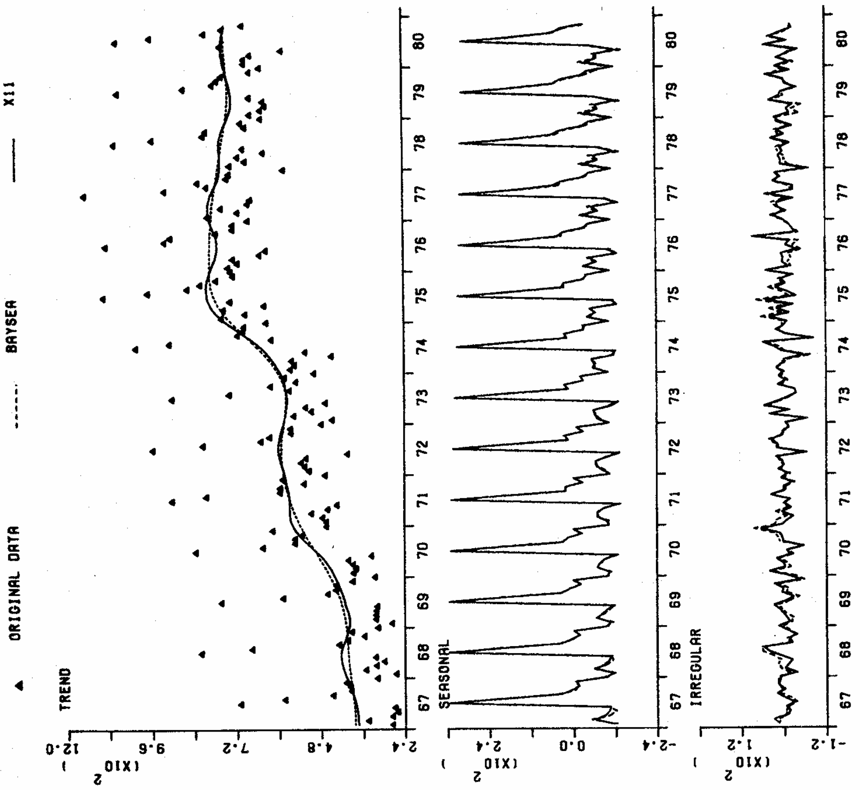


Figure 3. ALL EMPLOYEES IN FOOD INDUSTRIES
(Establishment data)

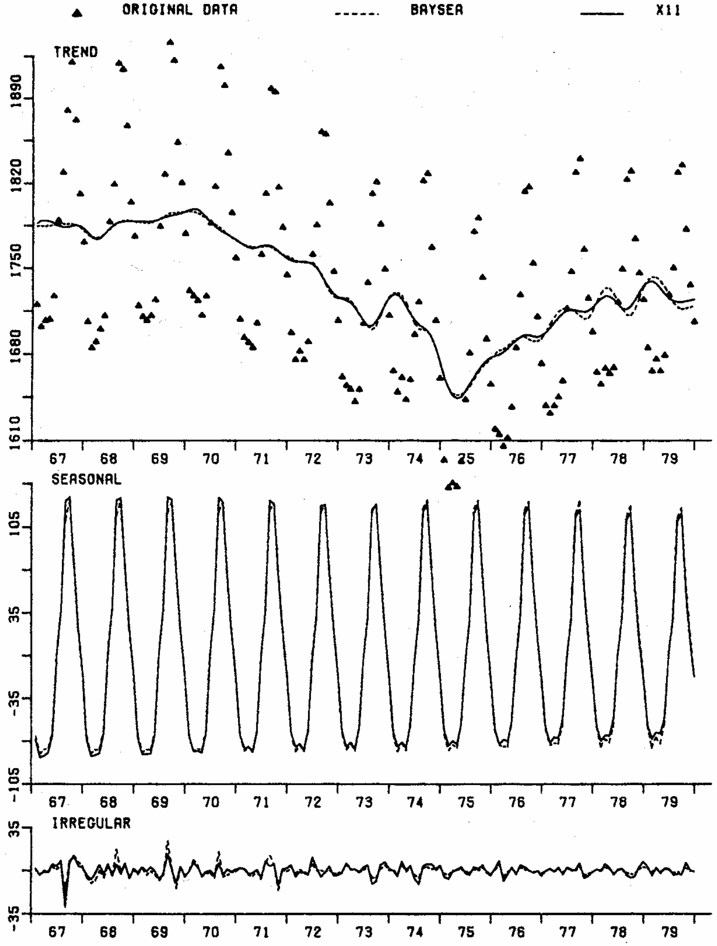


Figure 4. DEMAND DEPOSIT COMPONENT OF M-1A MONEY SUPPLY
(In log scale)

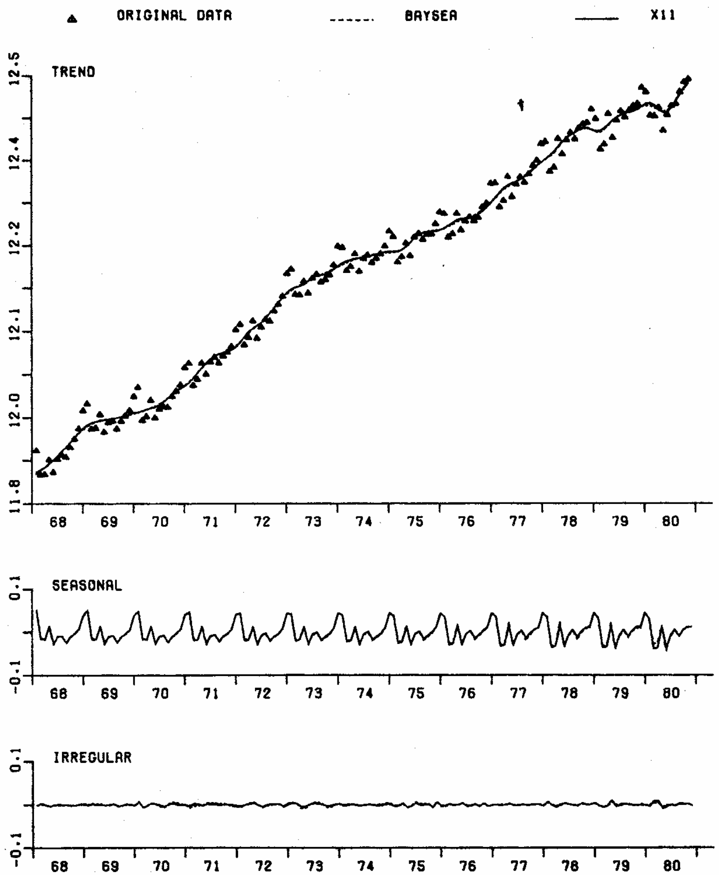


Figure 5. RETAIL SALES OF AUTOMOTIVE DEALERS
(In log scale)

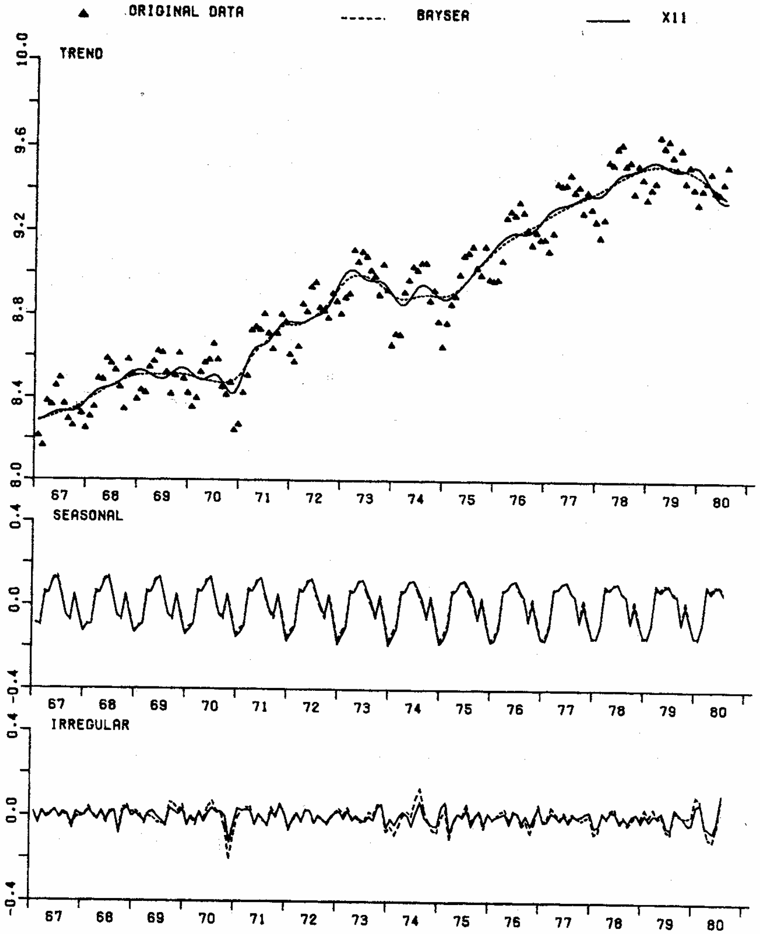


Figure 6. VALUE OF SHIPMENTS, BLAST FURNACE AND STEEL MILLS
(In log scale)

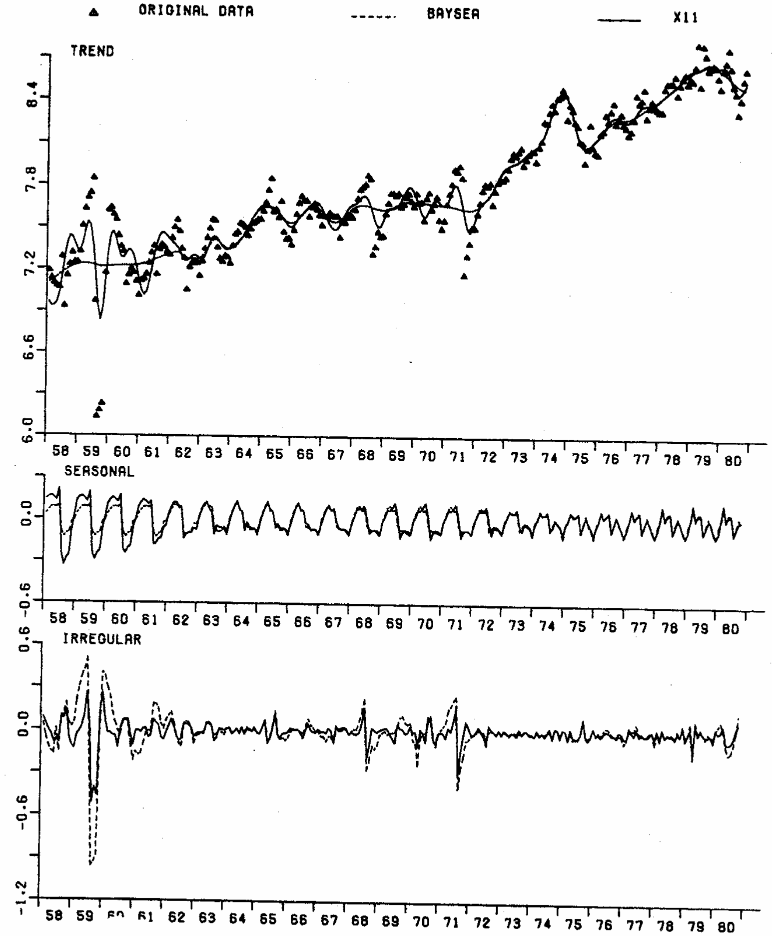


Figure 8. SPECTRA OF THE COMPONENT SERIES OF FIGURE 7

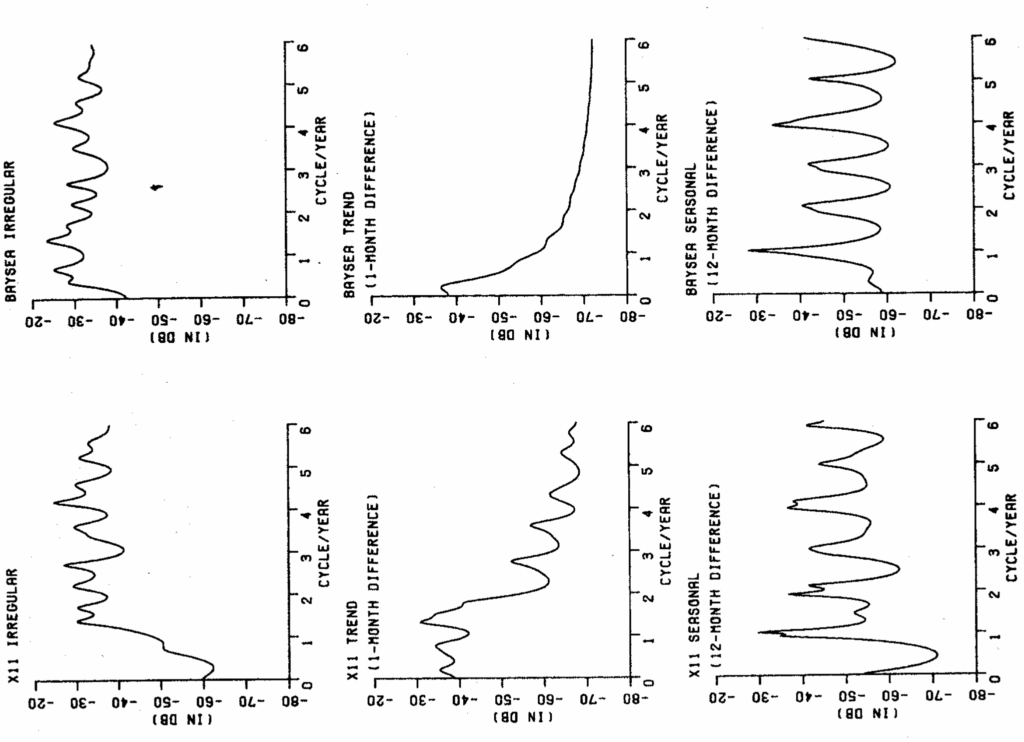


Figure 7. VALUE OF SHIPMENTS, BEVERAGES
(In log scale)

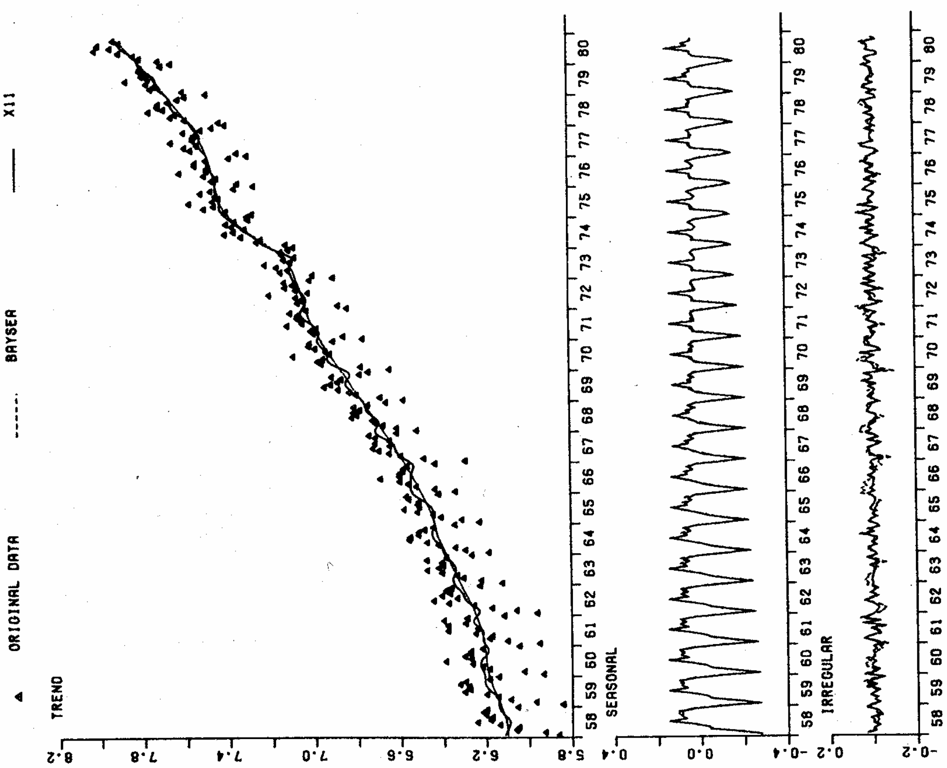


Figure 10. U.S. UNEMPLOYED, 16 YEARS OLD AND OVER

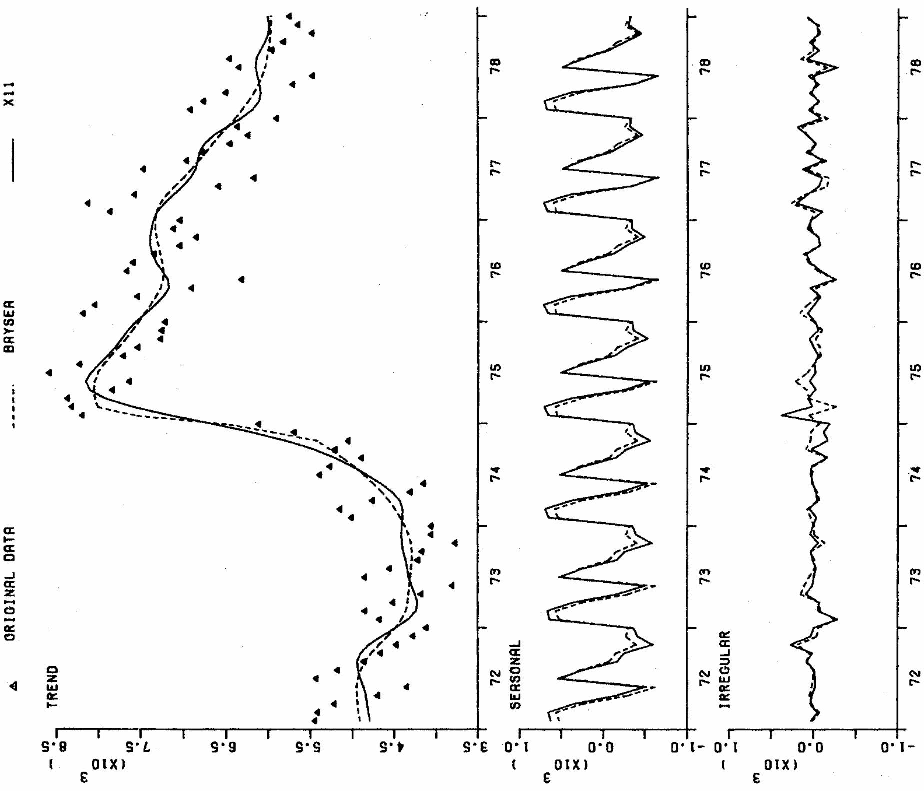


Figure 9. HOUSING STARTS, NORTH CENTRAL, DWELLINGS OF FIVE OR MORE FAMILIES
(In log scale)

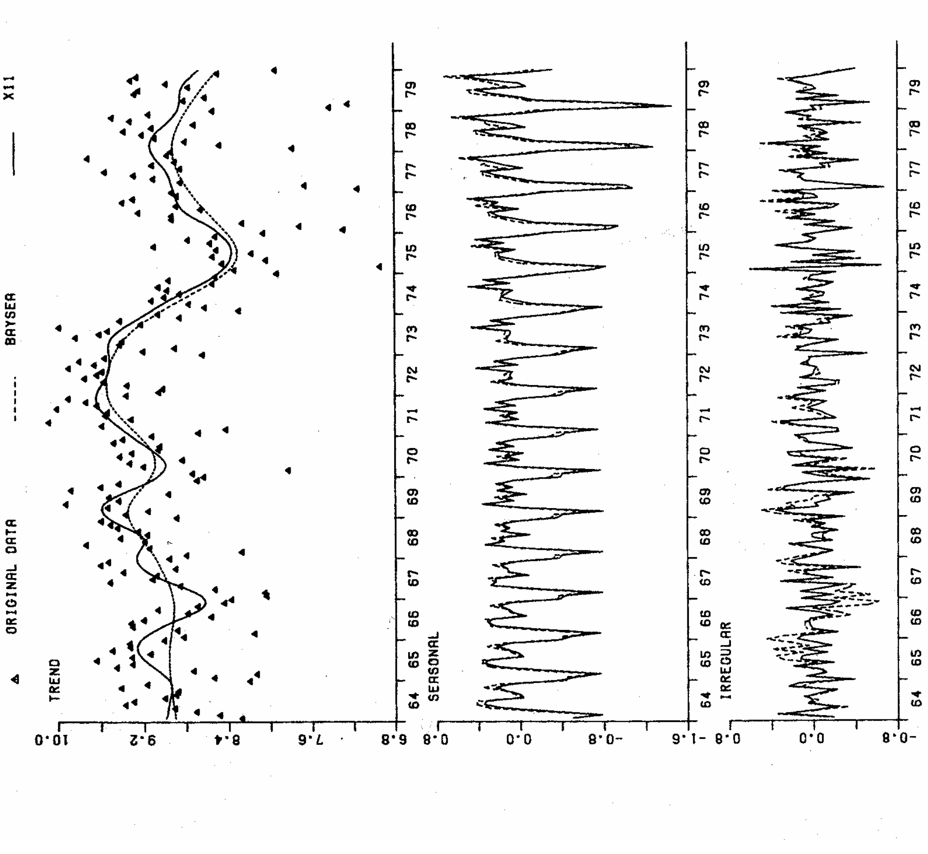
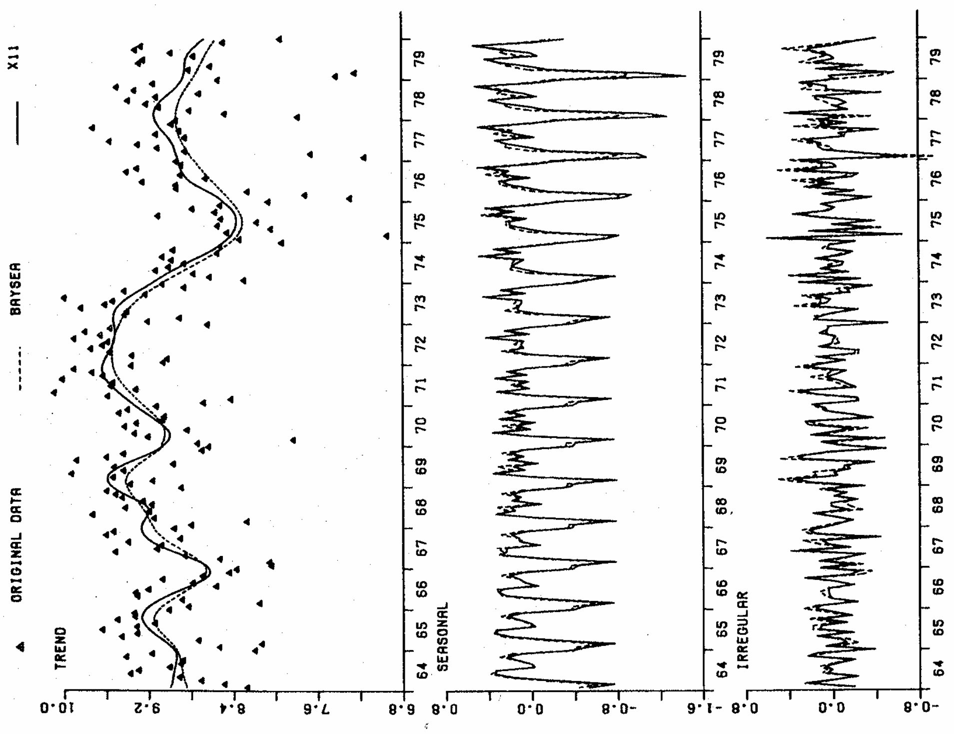


Figure 11. AVERAGE BAYSE ESTIMATES OF HOUSING STARTS



COMMENTS ON "COMPARATIVE STUDY OF THE X−11 AND BAYSEA PROCEDURES OF SEASONAL ADJUSTMENT" BY H. AKAIKE AND M. ISHIGURO

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Statistics Canada

INTRODUCTION

I have read with interest the paper by Akaike and Ishiguro, "Comparative Study of the X−11 and BAYSEA Procedures of Seasonal Adjustment." These authors introduce a Bayesian seasonal adjustment (BAYSEA) method where the selection of the optimal model is based on the minimization of a well-defined objective function. They compare the revisions of the various components as estimated by BAYSEA, X−11, and X−11 ARIMA for 14 economic time series and point out the advantages of the former over the two latter. My comments concentrate primarily on three important issues: (1) The basic assumption of BAYSEA, for a method is optimal in the measure that the series to which it applies does not depart from its basic assumptions; (2) the serious limitations of the empirical comparisons which invalidate the authors conclusions; and (3) the operational state of the BAYSEA program for use by a statistical agency.

1. BASIC ASSUMPTIONS OF THE BAYSEA METHOD

The Baysea method is based on the minimization of the following objective function:

$$\{[Y_t - T_t - S_t]^2 + d^2[s^2(\Delta^2 T_t)^2 + (\Delta_{12} S_t)^2 + z^2(\sum_{j=0}^{11} S_{t-j})^2]\} \quad (1)$$

where $\Delta = 1 - B$ is the ordinary difference operator and $\Delta_{12} = 1 - B^{12}$ is the seasonal difference operator. The terms which are squared in the first sum are the deviations of the observations from the estimated components and those in the second, the second order difference of the trend-cycle, the first difference of the seasonality and the constraint of the seasonal estimates to sum to zero over a year. The order of the difference in (1) corresponds to the default options of the computer program, but other orders may be chosen by the user depending on the characteristics of the series. s and z are constants to be chosen a priori by the analysts and they are related to the degree of smoothness of the trend-cycle and to the annual stability of the seasonality, respectively. The degree of flexibility of the seasonal pattern is given by $1/s$, where small values imply a more flexible seasonality.

The main contribution of Akaike and Ishiguro to the class of seasonal adjustment methods based on the minimization of an objective function as (1) is the optimal estimation of the scaling factor d which measures the degree of smoothness of the signal; the trend-cycle plus the seasonality.

The idea of applying least squares techniques to time series under the form of minimizing a linear combination of two sums of squares was first suggested by Henderson (1919) and later developed by Whittaker (1923, 1924) and Henderson (1924).

Henderson (1919) pointed out that in smoothing real data there is always a compromise between how good the fit should be and how smooth the fitted curve should be. The lack of fit is measured by the sum of squares of the deviations between observed and fitted values, and the lack of smoothness, by the sum of squares of the third differences of the smoothed curve. The Whittaker-Henderson method was originally used for trend fitting with weights calculated for the whole span of the series. The objective function to be minimized was of the following form

$$k\Sigma_t(Y_t - T_t)^2 + \Sigma_t(\Delta^3 T_t)^2 \quad (2)$$

The larger the k, the more importance was given to closeness of fitting versus smoothing. Equation (2) was later modified by Lesser (1961, 1963), Cholette (1978), and Schlicht (1981) to incorporate seasonal smoothing and to allow other differencing orders.

Contrary to the authors' claim that (p. 3): " . . .this approach leads to a regression technique with a very simple structure . . .," the BAYSEA method belongs to the same class of

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moving average techniques as the X-11 procedure. It differs from X-11 mainly in that the estimation of the components is made simultaneously and the weights are calculated by least squares. In X-11, the estimation is iterative and the weights are based on summation formulas developed by actuaries. (For the latter, see Macaulay 1931.)

The basic assumptions of the default option of BAYSEA are that the time series would tend to follow a straight line with a superimposed stable seasonality if unaffected by random shocks. The random shocks are thought to be partly of a permanent character and partly temporary. The permanent disturbances become incorporated in the trend-cycle and the seasonal estimates to change their level and/or direction, while the temporary disturbances leave both systematic components unaffected. The permanent disturbances are defined as the second differences of successive values of the trend-cycle plus the first seasonal differences of each month, the temporary disturbances as deviations of the observations from the fitted values. The amount of the random shock that will be incorporated permanently as part of the trend and seasonality depends on the value of d which is to be estimated by the program. The higher d is, the smoother the estimated trend-cycle and the seasonality.

The basic assumptions of BAYSEA are comparable with those of X−11 from the viewpoint of the model-based seasonal adjustment procedures. In fact, Cleveland and Tiao (1976) proved that the symmetric weights of the standard option of X−11 can be well approximated from the decomposition of an IMA model of the following form:

$$(1 - B)(1 - B^{12})Y_t = \theta(B)a_t \qquad (3)$$

and Tiao (1980) proved that the objective function of BAYSEA leads to the following overall model for the series

$$(1 - \phi B)(1 - B)(1 - B^{12})Y_t = \theta(B)a_t \qquad (4)$$

where the value of ϕ is fixed.

The moving average weights, however, are not the same for both methods. In BAYSEA, the weights change in function of the model chosen and the value of the parameter d. On the other hand, in the X−11 method, the weights change in function of the span of the seasonal and trend-cycle moving averages. The shorter the span of the moving average, the more the irregular variation of the series is absorbed into the trend-cycle and the seasonality.

Because the properties of moving average techniques are best analysed by looking at the transfer functions of the linear filters, the gain and phase shift functions of the trend-cycle and the seasonal filters of the default options of BAYSEA and X−11 have been calculated. For BAYSEA, the gain and phase functions are shown for $d = 1$, and $d = 4$, values often encountered when applying the standard option to the real series but calculations have also been done for $d = 10$, 12, 16 and 20. (d can take values from 1 to 20 in the BAYSEA program.)

Figure 1 shows that the gain of the central trend-cycle filter of BAYSEA for $d = 4$ suppresses more the short cyclical fluctuations as compared to the corresponding gain function of X−11 in figure 5. The gain of cycles of 36 to 18 months goes from 98 percent to 84 percent and, thus, series strongly affected by this type of cyclical fluctuations will be over-smoothed. On the other hand, the gain of the central trend-cycle filter for $d = 1$, in figure 3, indicates that a large variation of the irregular goes into the trend estimates.

Figures 2 and 4 show that there is practically no difference between the gain functions of the corresponding central seasonal filters of BAYSEA for $d = 4$ and $d = 1$, respectively. On the other hand, the concurrent seasonal filters are quite different, being the gain function of $d = 1$ closer to the central than that for $d = 4$. In both cases, however, the central and concurrent seasonal filters of BAYSEA differ significantly from those of X−11 shown in figure 6. The BAYSEA central and concurrent seasonal filter pass more noise variation as compared to X−11.

For other d values (not shown here) only the gain functions of the trend-cycle filters are affected, being more insensitive to short cycles as d increases. For $d = 20$, only 75 percent of the gain of the 36-month cycle is passed.

The phase shifts calculated for the concurrent filters are shown in table 1 for those values greater than 1 month at the frequencies where the gain is not close to zero. The results show that the phase shifts of the BAYSEA concurrent seasonal filters are larger than those of X−11, particularly, at the very important frequencies around the fundamental seasonal frequency.

Table 1. PHASE SHIFTS INTRODUCED BY THE CONCURRENT SEASONAL FILTERS OF THE X−11 AND BAYSEA METHODS

X−11 method		BAYSEA method		
Frequency (in degrees)	Phase shift (in degrees)	Frequency (in degrees)	Phase shift (in months) d=4	d=1
26	−3.4	26	3.3	−3.6
28	1.7	28	2.0	−1.8
32	−1.5	32	−2.0	2.8
34	2.4	34	−2.0	0
38	−1.3	36	1.9	1.9
40	−1.6	38	2.3	0
42	−1.4	40	1.7	−1.3
44	−1.6	42	1.5	0
48	−1.6	44	1.5	−1.5
50	−1.5	46	1.3	−1.0
52	1.3	48	1.2	−1.6
62	−1.1	50	1.1	−1.5
		52	0	−1.5
		54	0	1.5
		56	0	1.4
		64	0	1.3

0 = Phase shift smaller than 1 month.

Figure 1
Gain Functions of the Central and Concurrent Trend-cycle Filters of BAYSEA

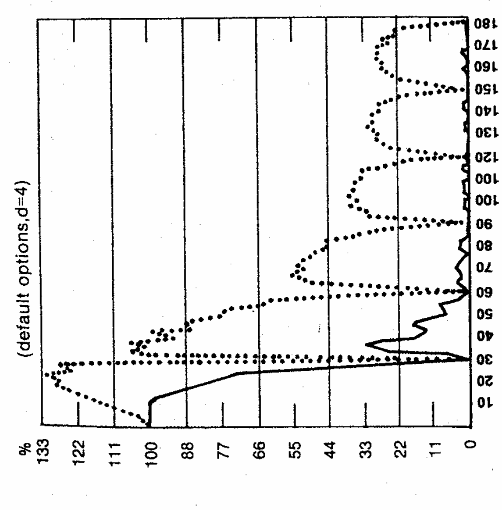


Figure 2
Gain Functions of the Central and Concurrent Seasonal Filters of BAYSEA

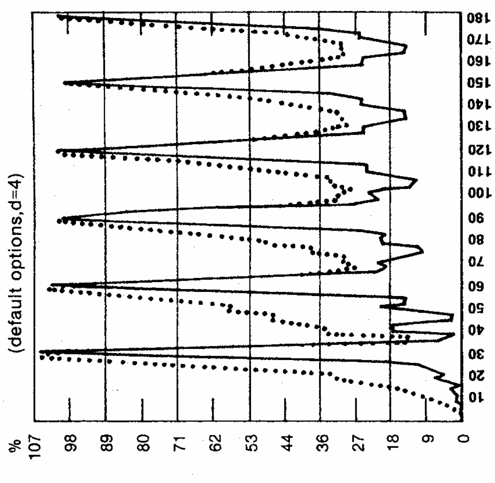


Figure 3
Gain Functions of the Central and Concurrent Trend-cycle Filters of BAYSEA

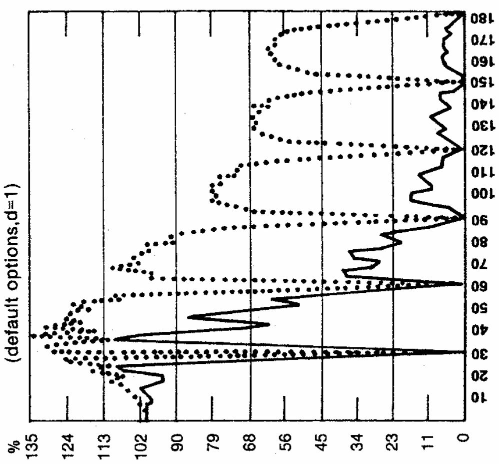


Figure 4
Gain Functions of the Central and Concurrent Seasonal Filters of BAYSEA

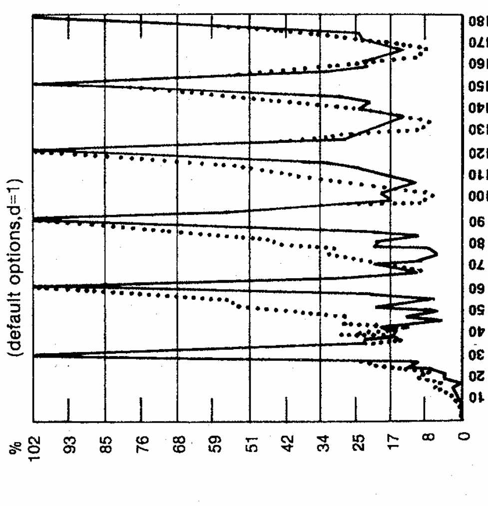


Figure 5
Gain Functions of the Central and Concurrent Trend-cycle Filters of X-11

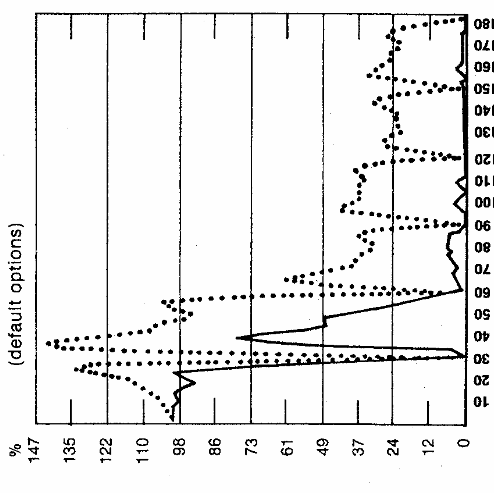
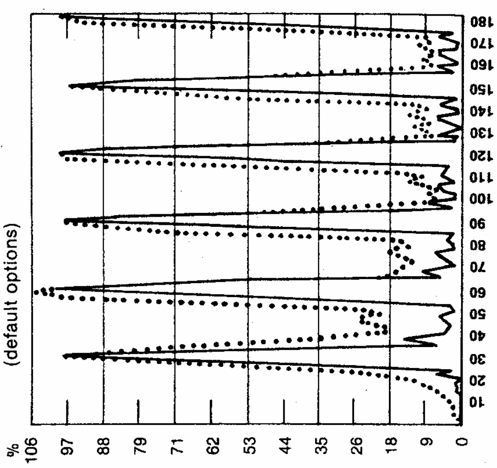


Figure 6
Gain Functions of the Central and Concurrent Seasonal Filters of X-11



——— Central filter .... Concurrent filter

2. LIMITATIONS OF THE EMPIRICAL COMPARISONS

The revisions of current seasonally adjusted values by moving average procedures are due to (1) differences in the properties of the linear filters applied to the same observation as it changes its time position and (2) the innovations that enter into the series with new observations.

The authors compare the revisions of the concurrent estimated values of the trend-cycle, the seasonality and the irregulars obtained with X−11, X−11 ARIMA, and BAYSEA for 14 series. The analysis is done without applying the option for replacement of outliers because (p. 14) "...since these options will be effective only when the basic procedures work satisfactorily to give good initial estimates of the irregular component, we limited our attention mainly to the comparison of the procedures without these options." It is apparent, however, that if the outliers are not removed, the estimates of the trend-cycle and seasonality by these two methods will be seriously distorted and their ranking according to the size of the revision is thus misleading. A further limitation that invalidates the conclusions of the comparison is the fact that the authors apply different BAYSEA models in function of the characteristics of the series whereas the option of X−11 and X−11 ARIMA are the same for all the series.

To measure the size of the revisions due to filter changes (ignoring the treatment of outliers option as done by the authors) it is more appropriate to calculate the mean absolute difference between the gain functions of the central and concurrent filters of each method. The results given in table 2 show that the revisions of the X−11 concurrent trend-cycle filter are smaller than those of BAYSEA for $d = 1$ and $d = 4$. Concerning the concurrent seasonal filter, the revisions of BAYSEA for $d = 1$ are the smallest. However, these revisions indicate the speed of convergence of each concurrent filter towards its corresponding central filter, and since the central filters of BAYSEA and X−11 are significantly different, the comparison of these methods from this viewpoint has no meaning at all.

The filter revisions of X−11 ARIMA are not given here because they change in function of the ARIMA extrapolation model and its parameter values. Dagum (1981) shows, however, that for the three built-in models of the program and various sets of parameter values, the revisions are around 30 percent to 50 percent smaller than those of X−11 and, a fortiori, those of BAYSEA.

Looking at figures 2, 4, and 6, it is clear that the troughs of the gain functions of the X−11 central seasonal filter are closer to zero as compared to those of BAYSEA which passes too much noise. If the revisions are taken with respect to the X−11 central seasonal filter, the results given in table 3 show that the revisions of the BAYSEA concurrent seasonal filters are now much larger than those of X−11 in both cases.

Table 2. MEAN ABSOLUTE DIFFERENCES OF THE GAIN FUNCTIONS OF THE CONCURRENT TREND-CYCLE AND SEASONAL FILTERS OF VARIOUS METHODS WITH RESPECT TO THEIR CORRESPONDING CENTRAL FILTERS

Method	M.A.D. (concurrent trend-cycle filter)	M.A.D (concurrent seasonal filter)
Census X−11 (default options)	31.4	15.49
BAYSEA (default options, d=4)	35.11	18.95
BAYSEA (default options, d=1)	45.42	12.44

Table 3. MEAN ABSOLUTE DIFFERENCES OF THE GAIN FUNCTIONS OF THE CONCURRENT SEASONAL FILTERS OF BAYSEA AND X−11 WITH RESPECT TO THE CENTRAL SEASONAL FILTER OF X−11

Method	M.A.D. (concurrent seasonal filter)
Census X−11 (default options)	15.49
BAYSEA (default options, d=4)	29.65
BAYSEA (default options, d=1)	19.25

The main conclusions we can draw from this discussion is that the comparison of revisions of seasonal adjustment methods is useless if the methods have very dissimilar central filters. We must first decide on which is the optimal central filter according to some well-defined criteria.

3. OPERATIONAL STATE OF THE BAYSEA PROGRAM

Seasonal adjustment methods used by government statistical agencies must fulfill, besides well-accepted statistical criteria, other operational requirements as ease of interpretation and low operating costs.

Concerning the first operational condition, it is difficult to decide when the BAYSEA model chosen is, in fact, the best for the series in question. In our experimentation, we used the default options to start, and if no messages were printed by the program indicating that d was reaching its minimum, the model was considered adequate. However, the authors suggest to choose the model that gives the smallest ABIC which means that several "order," "sorder," and "rigid" must be tried before making a decision. Table 4 shows the

ABIC and d values for the agricultural employment of males 20 years old and over for various models.

Table 4. ABIC AND *d* VALUES FOR AGRICULTURAL EMPLOYMENT OF MALES 20 YEARS OLD AND OLDER

BAYSEA model	ABIC	d
(2, 2, 1.0)	386,783	4.2
(2, 1, 1.0)	387,966	2.5
(1, 1, 1.0)	374,096	1.49
(1, 1, .5)	375,141	1.49

The authors have chosen the (1, 1, 1.0) model which corresponds to the smallest ABIC, but the difference with that of the (1, 1, .5) model is so negligible that any user will doubt on which is to be preferred. These two models, however, imply different behaviour of the seasonal component being more stochastic for the second model than for the first. The same would happen if one would have to decide between the (2, 1, 1.0) and (2, 2, 1.0) models. Furthermore, when we experimented with the (1, 1, 1.0) model, the program printed that a higher order and/or sorder must be tried, but when we did so, the estimated ABIC's as shown in table 4 were also slightly higher.

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COMMENTS ON "COMPARATIVE STUDY OF THE X−11 AND BAYSEA PROCEDURES OF SEASONAL ADJUSTMENT" BY H. AKAIKE AND M. ISHIGURO.

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The BAYSEA seasonal adjustment procedure (Akaike 1980) uses an estimated expected log maximum likelihood procedure to select the best model from among several classes of simple transformed-Gaussian smoothing models for a given seasonal time series. Akaike and others have provided a substantial amount of evidence that such procedures can be effective in a variety of statistical contexts (see Akaike 1977 for a number of examples) when the models being considered are reasonable ones for the data. The good results he and Ishiguro report in the paper under discussion suggest that their seasonal models are adequate for the adjustment of many seasonal economic time series. Our comments will report results related to some issues not directly addressed by their study: (1) The comparative accuracy of the BAYSEA and X−11 adjustments on some synthetic seasonal time series, (2) the seasonal adjustment by BAYSEA of some short series, (3) its performance in selecting pre-adjustment transformations of the data, (4) BAYSEA's relationship to ARIMA model-based optimal smoothing procedures for seasonal adjustment. We are indebted to George Tiao for kindly permitting us to append his study notes on this topic to these comments.

BAYSEA does well in the limited studies done by us to investigate (1)−(3). Further evaluation work will be required to reach firmer conclusions. Our experience suggests that the use of synthetic series of the sort described in section 1 below can and should play a significant role in such evaluations.

For all series adjusted by BAYSEA in this paper, the adjustment which gave the smallest among three values of AVABIC was used: These values of AVABIC were obtained with RIGID=1 by allowing (ORDER, SORDER) to take on the values (1,1), (2,1) and (2,2). A more careful choice of these parameters could be expected to yield further improvements.

The measure of smoothness described by the average absolute percentage change (AAPC) for months that are a fixed lag apart is one of the criteria used to evaluate the effectiveness of a seasonal adjustment. For a series $x_1, \ldots, x_N$ of length N and for a given lag, k, the lag k version of this criterion is calculated by means of

$$\sum_{t=k+1}^{N} |x_t - x_{t-k}| / |x_{t-k}|$$

These quantities appear in table F.2 of the X−11 program output. It is considered desirable that such numbers be smaller for the adjusted series than for the unadjusted series and, often but not always, the smoother of two adjustments is the better one. We shall use this criterion for comparison purposes.

1. ACCURACY OF THE SEASONAL ADJUSTMENTS FOR SOME SYNTHESIZED SERIES

In the report by Hillmer and Bell (1980), seasonal adjustments for 76 series were obtained by an ARIMA model-based signal extraction procedure and were compared with the default-option adjustments obtained from X−11. Often, both procedures produce quite similar seasonals. This is the case for the series 503WL and 518WL, which are identified in table 1.1. The seasonals from both methods for these series are depicted in figures 1.1 and 1.2. It seems reasonable, therefore, to accept these numbers as seasonals. The trend obtained from the ARIMA model-based adjustment of a third series, TI506 (table 1.1 and figure 1.3), the X−11 seasonals from 503WL and 518WL, and two "irregular" series were combined in various ways to form a set of synthetic series to be used for comparing the accuracy of the adjustments obtained by BAYSEA and by X−11. The correct seasonal adjustment is defined in each case to be the trend times the irregular. Denoting this correctly adjusted series by $x_1, \ldots, x_N$ and the estimates obtained from one of the methods by $\hat{x}_1, \ldots, \hat{x}_N$, the relative accuracy of the estimates was assessed by means of two measures, RRMSQD (relative root mean squared deviation) and RMAD (relative mean absolute deviation), defined as follows:

$$\mathrm{RRMSQD} = [N^{-1}\sum_{t=1}^{N} (x_t - \hat{x}_t)^2/x_t^2]^{1/2}$$

$$\mathrm{RMAD} = N^{-1}\sum_{t=1}^{N} |x_t - \hat{x}_t| / |x_t|.$$

The components used to synthesize the series are described in table 1.2. The numerical values of T, S_1 and S_2 are given in table 1.3. The descriptions in terms of components of the synthesized series, along with the values of the accuracy measures associated with the default-option X−11 and BAYSEA seasonal adjustments, are presented in table 1.4.

Figure 1.1 X−11 (solid) **AND ARIMA MODEL-BASED** (dashed) **SEASONALS FOR 503 WL** (from Hillmer and Bell 1980)

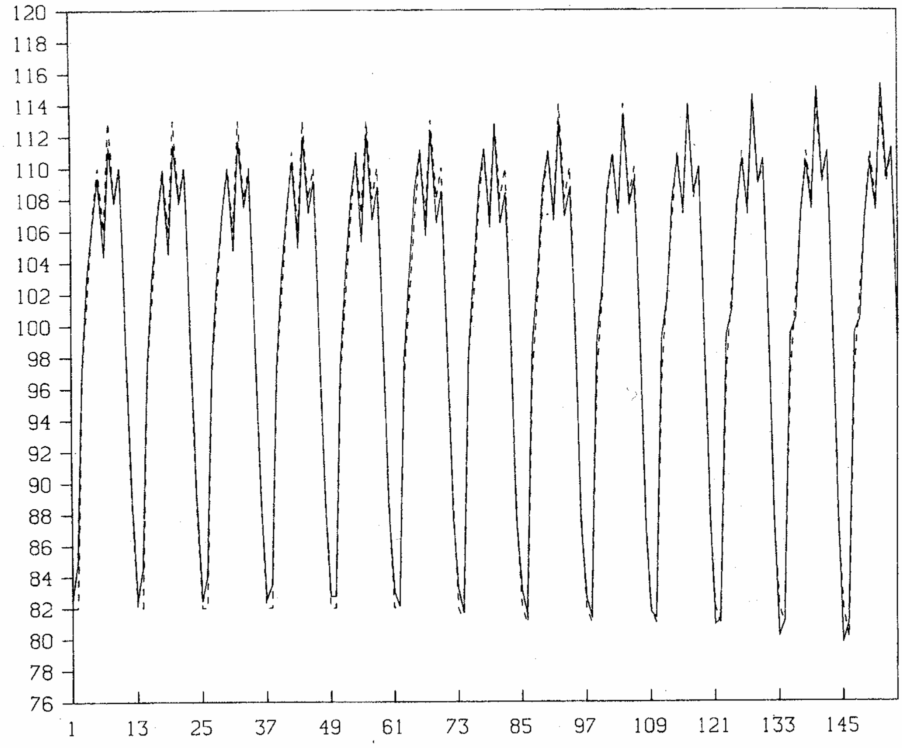


Figure 1.2 X−11 (solid) **AND ARIMA MODEL-BASED** (dashed) **SEASONALS FOR 518 WL** (from Hillmer and Bell 1980)

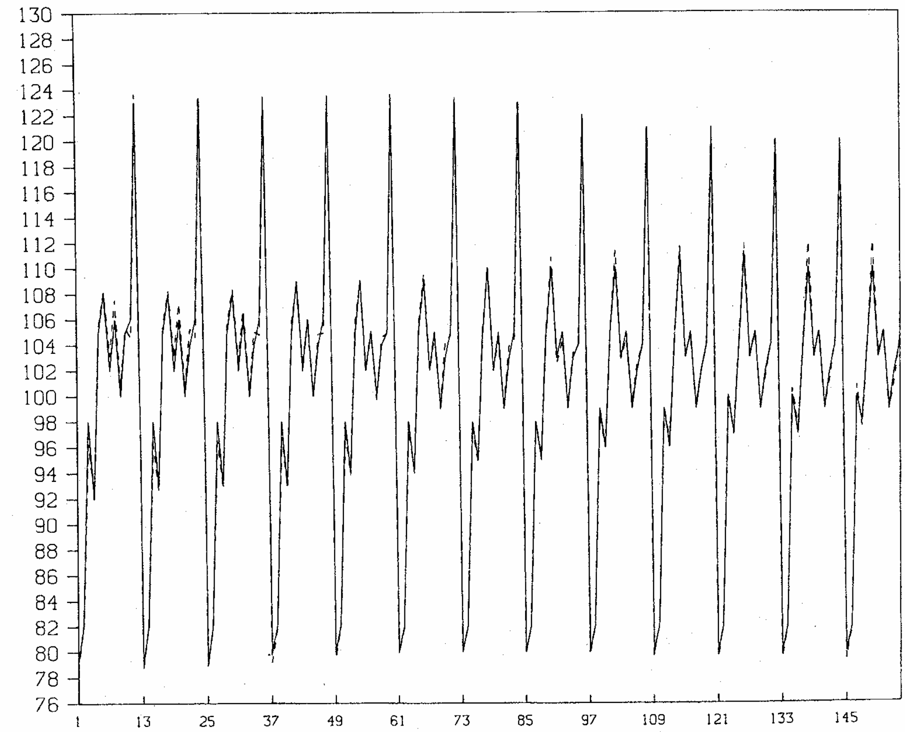


Figure 1.3 TREND T FROM T1506

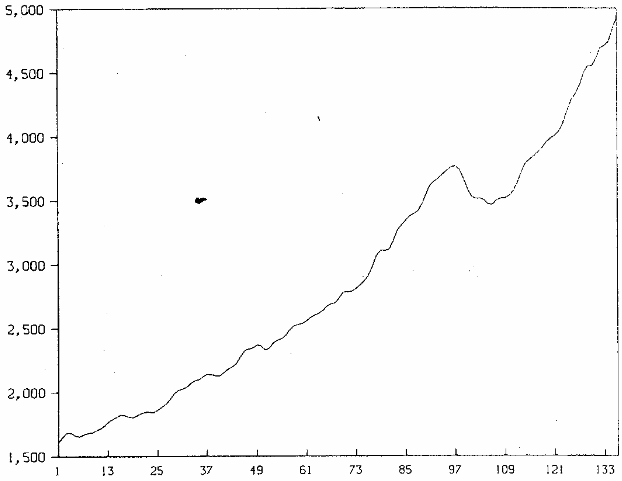


Table 1.1 IDENTIFICATION OF SERIES USED

TI506	Wholesale inventories—electrical goods
503WL	Wholesale sales—lumber and other building materials
518WL	Wholesale sales—beer, wine, and distilled acloholic beverages

Table 1.2 IDENTIFICATION OF COMPONENTS USED TO SYNTHESIZE SERIES

T	ARIMA model-based trend from TI506
S_1	X−11 seasonal from 503WL
S_2	X−11 seasonal from 518WL
I_1	$I_{1,t}$: $\log I_{1,t} \sim i.i.d\ N(0, 0.02)$
I_2	$I_{2,t}$: $\log I_{2,t} = (I - .66B)^6 \log I_{1,t}$

Table 1.3 TREND AND SEASONALS: T, S_1, S_2

	1610	1649	1685	1685	1660	1652	1669	1680	1685	1698	1714	1737
	1767	1787	1805	1822	1818	1805	1803	1817	1835	1845	1844	1841
	1861	1885	1906	1945	1988	2016	2025	2039	2069	2089	2098	2121
	2140	2137	2126	2128	2155	2182	2196	2225	2279	2324	2336	2345
	2369	2361	2328	2346	2388	2405	2417	2445	2487	2516	2526	2534
T	2555	2582	2599	2614	2637	2670	2690	2696	2738	2782	2783	2786
	2808	2836	2867	2908	2981	3067	3108	3104	3113	3176	3254	3302
	3338	3373	3390	3415	3469	3545	3619	3649	3669	3702	3733	3757
	3763	3738	3672	3585	3525	3509	3511	3496	3463	3464	3497	3512
	3513	3527	3568	3633	3714	3785	3817	3840	3871	3907	3951	3979
	3999	4036	4097	4194	4284	4332	4395	4500	4549	4547	4608	4691
	4706	4739	4842	4928								
	82.5	84.7	97.1	103.3	106.6	109.4	104.4	111.3	107.8	110.0	99.3	89.6
	82.4	84.5	97.1	103.2	106.7	109.6	104.6	111.5	107.7	109.9	99.3	89.5
	82.4	84.0	97.3	103.2	107.0	109.9	104.8	111.7	107.5	109.6	99.2	89.3
	82.6	83.5	97.5	103.3	107.4	110.4	104.9	111.9	107.1	109.1	99.1	89.0
	82.8	82.7	97.8	103.5	108.0	110.7	105.3	112.1	106.7	108.7	98.9	88.6
S_1	83.1	82.1	98.1	103.7	108.5	111.1	105.7	112.3	106.5	108.5	98.6	88.1
	83.3	81.7	98.3	103.6	108.9	111.2	106.2	112.6	106.5	108.4	98.5	87.7
	83.1	81.6	98.7	103.2	108.8	111.0	106.6	112.9	106.9	108.7	98.6	87.5
	82.6	81.5	99.0	102.5	108.6	110.7	107.0	113.3	107.6	109.2	98.8	87.6
	81.7	81.3	99.2	101.6	108.1	110.6	107.3	113.9	108.4	110.0	98.9	87.9
	80.9	81.2	99.3	100.9	107.6	110.5	107.4	114.6	109.0	110.5	99.0	88.2
	80.2	81.1	99.4	100.4								
	78.6	82.1	96.2	92.5	105.6	108.2	103.2	107.6	100.8	105.4	104.6	123.7
	78.7	82.0	96.2	92.6	105.6	108.3	103.1	107.3	100.6	105.3	104.6	123.6
	78.9	81.9	96.5	92.9	105.7	108.4	102.8	106.8	100.4	105.1	104.8	123.6
	79.2	81.8	96.8	93.3	105.8	108.7	102.4	106.0	100.0	104.9	104.9	123.7
	79.5	81.8	97.1	93.9	105.8	109.1	102.1	105.1	99.7	104.6	104.9	123.8
S_2	79.8	81.8	97.2	94.3	105.7	109.6	102.1	104.6	99.3	104.3	104.7	123.5
	80.0	81.9	97.5	94.8	105.4	110.2	102.3	104.4	99.2	103.9	104.4	122.8
	79.9	82.0	97.8	95.3	105.3	110.9	102.5	104.4	99.1	103.3	104.1	121.9
	79.7	82.1	98.4	96.0	104.9	111.3	102.9	104.4	99.3	102.6	103.9	120.7
	79.6	82.1	99.1	96.6	104.6	111.7	103.1	104.7	99.2	102.0	103.9	119.6
	79.5	82.1	99.9	97.0	104.3	111.9	103.2	104.9	99.2	101.6	104.2	118.8
	79.4	82.0	100.5	97.4								

Table 1.4 RELATIVE ERROR MEASURES FOR X−11 and BAYSEA

	RRMSQD (RMAD)	
	X−11	BAYSEA
$T \cdot S_1$	.009 (.007)	.006 (.004)
$T \cdot S_2$	.009 (.007)	.007 (.006)
$T \cdot S_1 \cdot I_1$	.084 (.069)	.051 (.042)
$T \cdot S_1 \cdot I_2$	.050 (.039)	.028 (.023)
$T \cdot S_2 \cdot I_1$	.081 (.068)	.052 (.044)
$T \cdot S_2 \cdot I_2$	.028 (.024)	.027 (.023)

Table 1.5 AVERAGE ABSOLUTE PERCENT CHANGES FOR $T \cdot S_2 \cdot I_2$

Lag	Original	X−11	BAYSEA
1	11.43	4.12	3.64
2	17.22	7.31	6.82
3	19.00	9.81	9.42
4	19.90	11.26	11.12
5	19.77	12.39	12.27
6	20.00	12.86	12.76
7	21.27	12.93	12.93
8	21.97	13.27	13.29
9	21.68	13.58	13.65
10	21.61	14.25	14.19
11	20.05	15.03	14.90
12	16.11	16.11	15.83

The effect of the two different kinds of irregulars, edited-lognormal white noise and an edited-lognormal sixth-order moving average process, are illustrated in figures 1.4 and 1.5. The editing was done to remove values larger than 1.5 standard deviations in magnitude from the associated pseudo-Gaussian series. Without editing, irregulars so obtained had implausible looking "outliers." (This suggests the possibility that the distributions of many economic time series have "thinner" tails than the lognormal.)

Graphs comparing the estimated and correct seasonally adjusted values for two of the series are given in figures 1.6−1.9. Since the statistics assessing accuracy give similar results for X−11 and BAYSEA with the series $T \cdot S_2 \cdot I_2$, the AAPC values for both adjustments of this series are also presented, in table 1.5.

Table 1.4 shows that even when BAYSEA's choice of models has been restricted to three classes, it is able to better remove standard-option X−11 seasonals from these series than is standard-option X−11. These seasonals are rather stable, however, and an experienced user of X−11 who knew this would probably use the nonstandard 3 × 9 filter option, which does give superior performance with these series. We have not yet compared finely tuned adjustments by X−11 and BAYSEA, since we lack the required experience with BAYSEA.

Figure 1.4 $T \times S_2$ (solid) **COMPARED WITH** $T \times S_2 \times I_1$ (dashed)

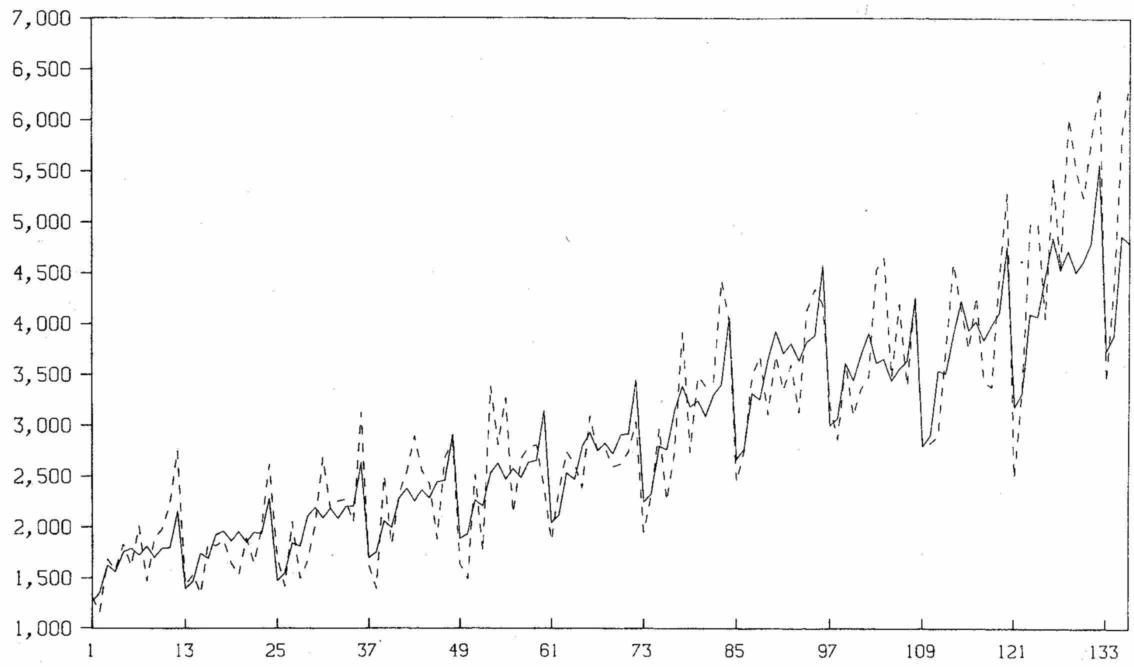


Figure 1.5 $T \times S_2$ (solid) **COMPARED WITH** $T \times S_2 \times I_2$ (dashed)

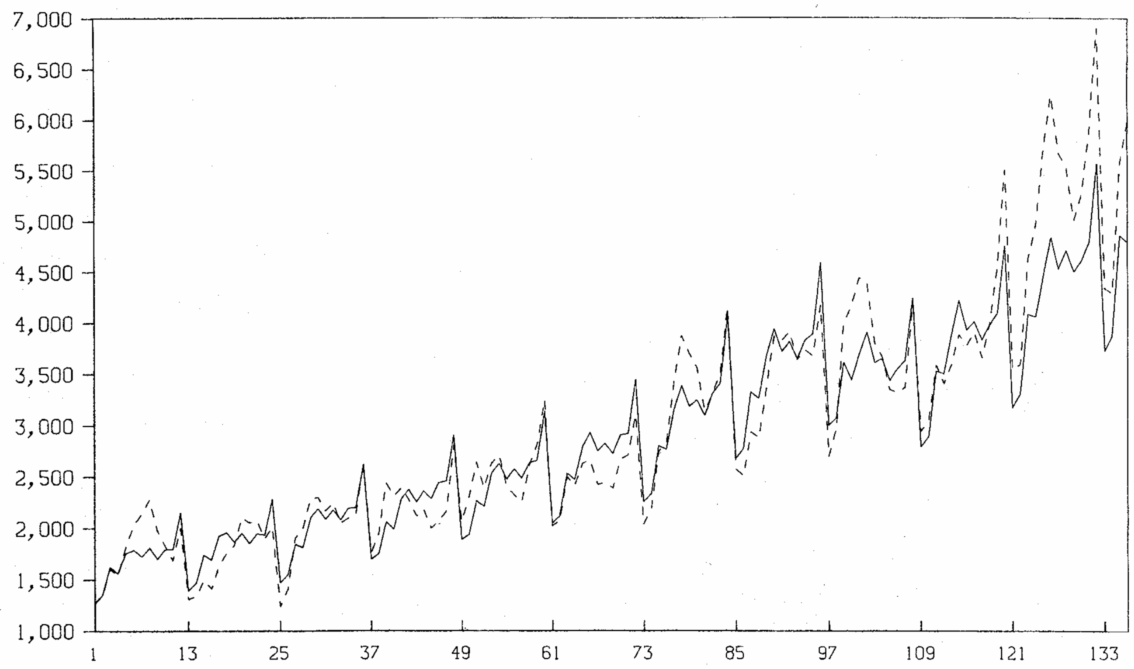


Figure 1.6 $T \times I_1$ (solid) **COMPARED WITH THE X–11 ADJUSTMENT OF** $T \times S_2 \times I_1$ (dashed)

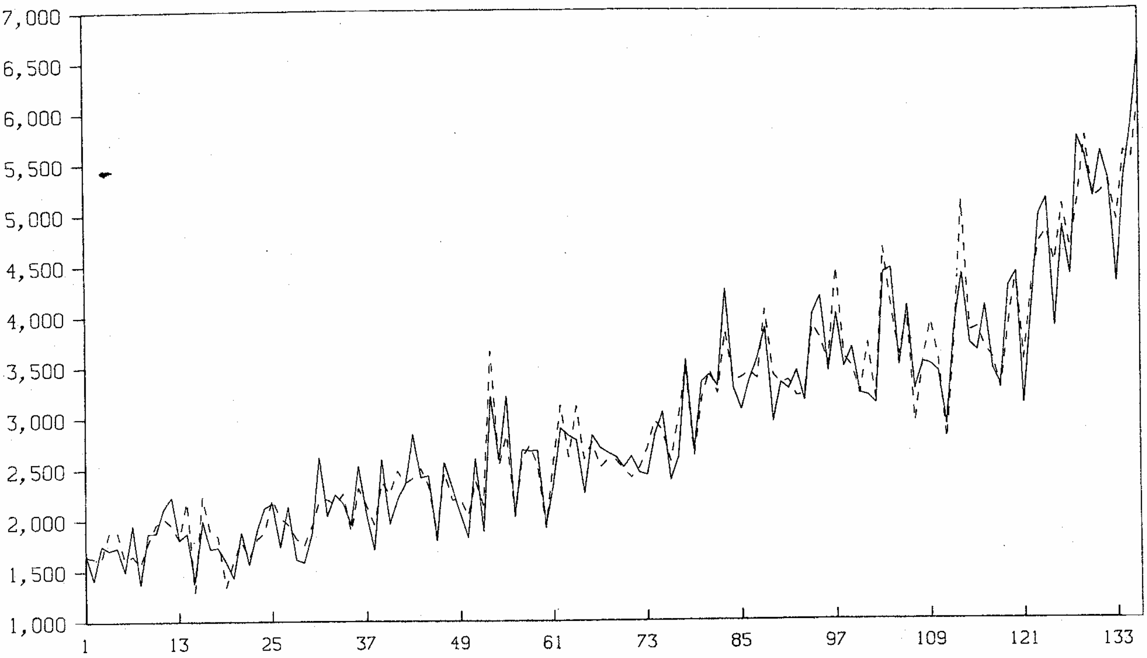


Figure 1.7 $T \times I_1$ (solid) **COMPARED WITH THE BAYSEA ADJUSTMENT OF** $T \times S_2 \times I_1$ (dashed)

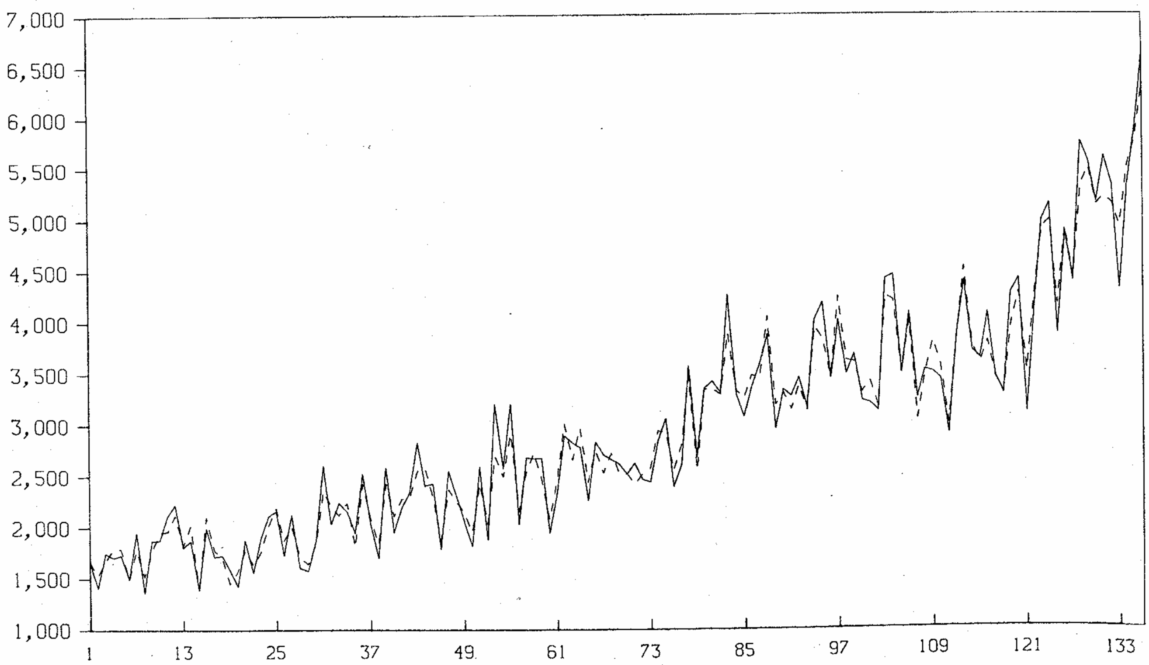


Figure 1.8 $T \times I_2$ (solid) **COMPARED WITH THE X–11 ADJUSTMENT OF** $T \times S_2 \times I_1$ (dashed)

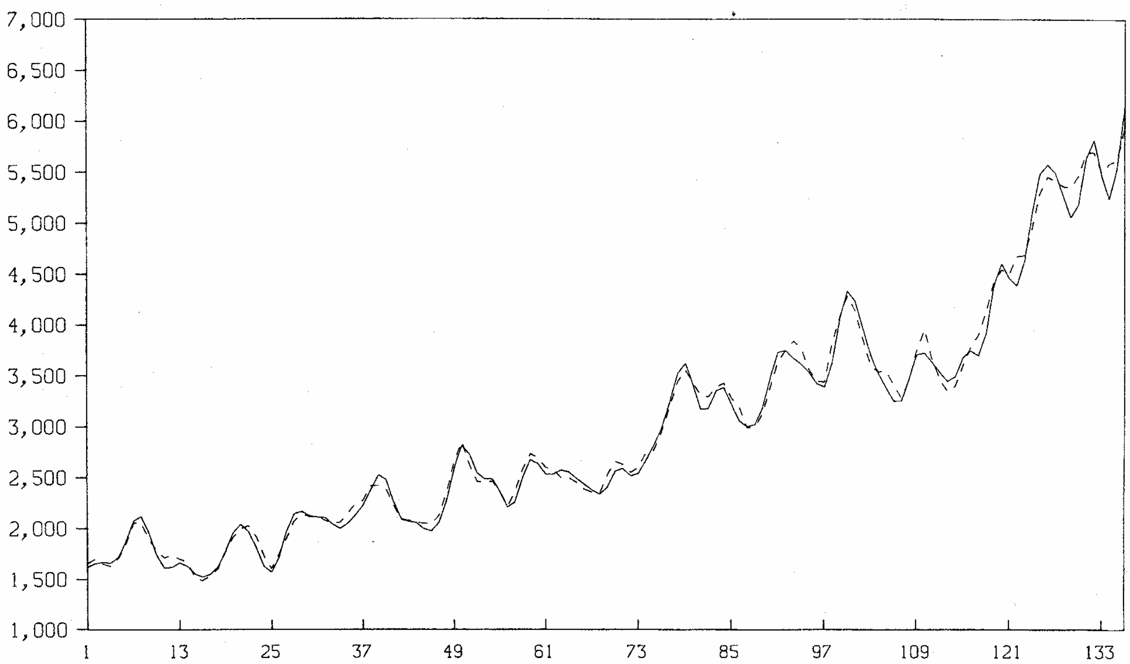
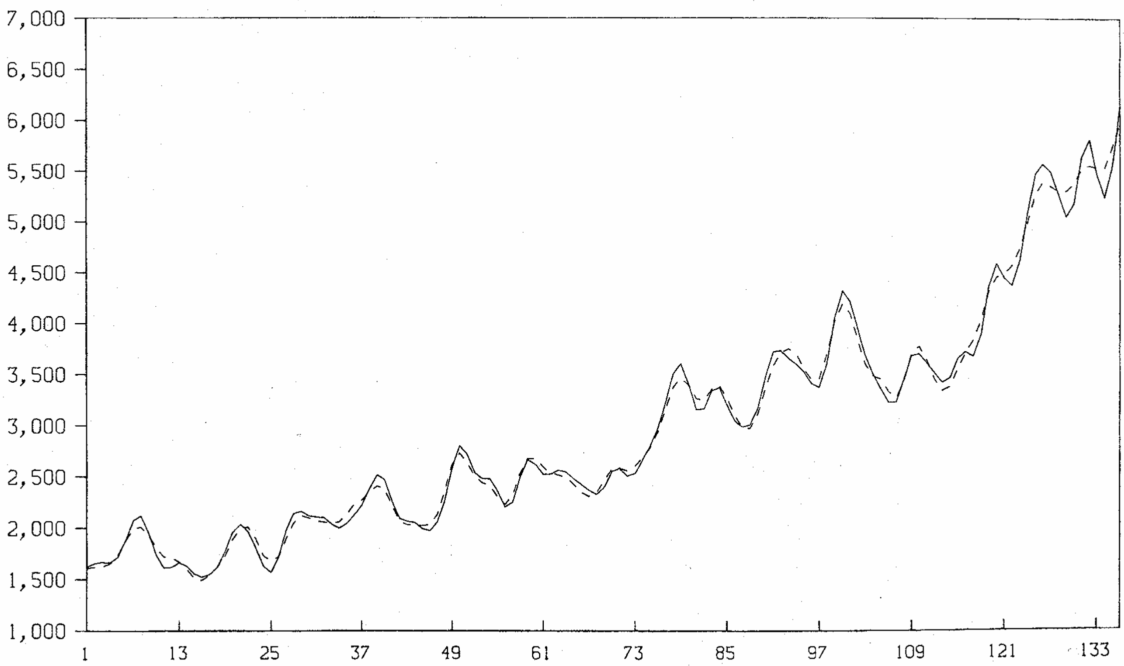


Figure 1.9 $T \times I_2$ (solid) **COMPARED WITH THE BAYSEA ADJUSTMENT OF** $T \times S_2 \times I_1$ (dashed)



2. ADJUSTMENT OF SHORT SERIES

Akaike's and Ishiguro's successful use of overlapping 4-year spans of data to accomplish adjustments by BAYSEA suggests that BAYSEA might be able to adequately adjust short time series. (When enough data are available, BAYSEA uses 7 years of data to backcast starting values for the trend and seasonal components and thereafter uses 4-year spans.) Five confidential defense unfilled orders series collected by the Census Bureau were strongly affected by the change in the beginning and ending of dates of the Federal Government's fiscal year, which occurred in 1976. Data for 51 months were available for these series, starting with October of 1976 when the first of the new fiscal years began. BAYSEA adjustments were made for all five of these series. None of the series was strongly seasonal. Three were thought to have negligible seasonality. For these three, BAYSEA produced seasonal factors very close to 1.0, and the average absolute percentage changes at most lags were higher for the adjusted series than for the unadjusted series. For the other two, some smoothing occurred and it was felt by subject-area experts that the adjustment was effective. Table 2.1 gives the AAPC values for the most seasonal of these two series, for its X−11 adjustment, and for its adjustment by BAYSEA. (The Census Bureau does not recommend the use of X−11 with such short series.)

Table 2.1 AVERAGE ABSOLUTE PERCENT CHANGES FOR A 51-MONTH, MODERATELY SEASONAL SERIES

Lag	Original	X−11	BAYSEA
1	3.31	2.87	2.38
2	5.18	4.22	3.75
3	6.78	5.44	5.11
4	8.08	6.73	6.42
5	8.87	7.96	7.62
6	9.64	9.25	8.66
7	10.19	9.89	9.69
8	11.42	10.96	10.76
9	12.48	11.98	11.76
10	13.23	12.89	12.60
11	13.93	13.82	13.61
12	14.66	14.66	14.56

3. TRANSFORMATIONS OF THE DATA

BAYSEA was applied to two Census Bureau series for which some evidence exists that a multiplicative (or log-additive) seasonal model is inappropriate. For the construction statistics series CON-BPNE1, which describes the number of building permits issued per month in the Northeast for single-family houses (figure 3.1) the value of AVABIC for an additive model was 1895 as contrasted with 1909 for the log-additive model. Thus, BAYSEA prefers the additive model for this series over the multiplicative one, which is a decision supported by visual and by some subject-matter considerations.

The AAPC values are given for the original data of CON-BPNE1, its additively adjusted series and its log-additive adjusted series (by BAYSEA in both cases) in table 3.1 below.

Table 3.1 AVERAGE ABSOLUTE PERCENT CHANGES FOR CON-BPNE1: 1/1969-3/1980

Lag	Original	Additive	Log-additive
1	22.76	5.30	5.79
2	40.62	7.39	7.87
3	50.78	8.73	9.49
4	55.34	10.38	11.20
5	56.84	11.35	12.37
6	56.28	11.35	14.20
7	55.02	13.98	15.30
8	52.46	15.36	16.98
9	47.40	16.44	17.87
10	39.93	17.69	19.01
11	28.92	18.61	19.90
12	21.71	19.74	21.02

The other series for which we considered various transformations was retail hardware sales. Denoting this series by y_t, we considered additive decompositions of y_t, $y_t^{.25}$ and $\log y_t$. BAYSEA (easily modified to consider such power transformations) preferred the additive decomposition of $y_t^{.25}$, as did an ARIMA model-based likelihood analysis performed by William Bell, which suggested the use of this transformation. The AAPC smoothness criterion favored the additive decomposition adjustments obtained from $y_t^{.25}$ and $\log y_t$ over that from y_t, but did not indicate a clear distinction between the adjustments obtained via $y_t^{.25}$ and $\log y_t$.

4. CONNECTIONS WITH ARIMA MODEL-BASED SEASONAL ADJUSTMENT

Suppose a trend-seasonal-irregular decomposition into uncorrelated components of the seasonal time series y_t is given by

$$y_t = T_t + S_t + I_t \tag{4.1}$$

With B denoting the backshift operator, let us make the additional assumption that T_t and S_t satisfy the difference equations

$$\phi_T(B)T_t = U_t \tag{4.2}$$

$$\phi_S(B)S_t = V_t \tag{4.3}$$

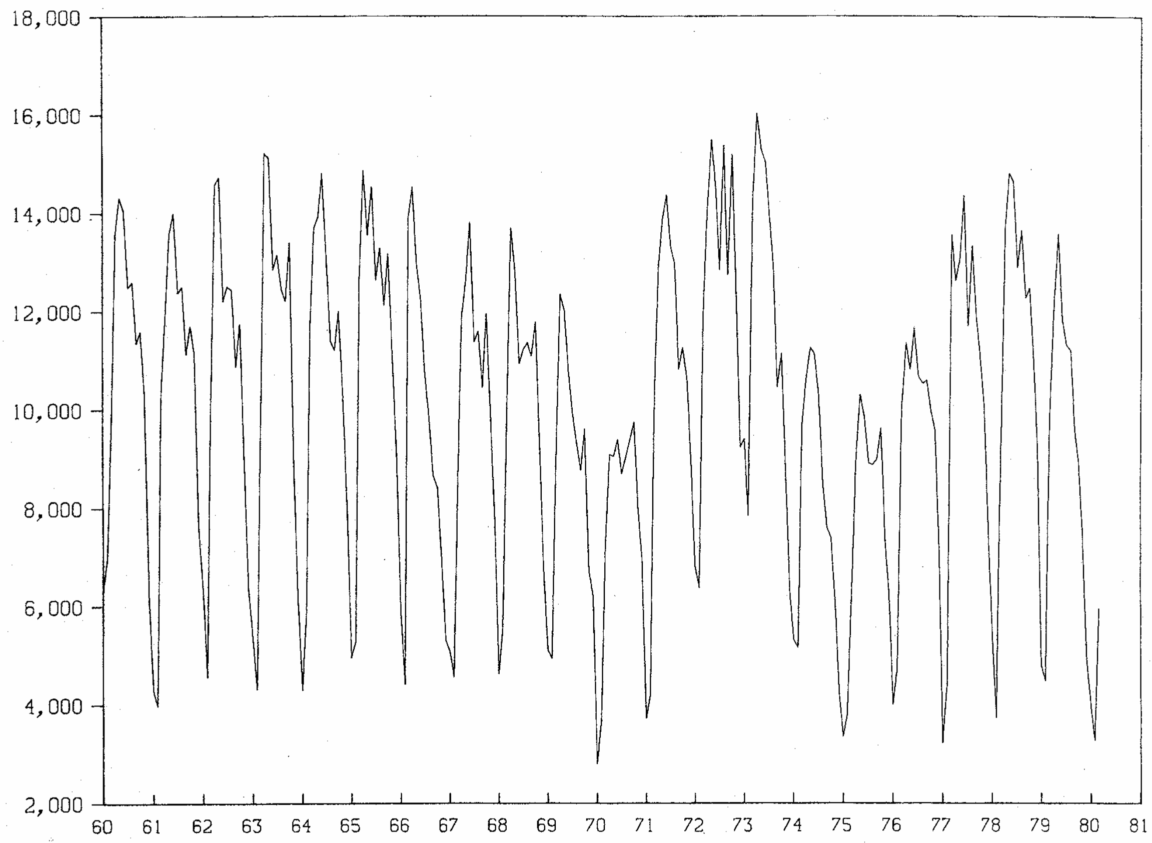
where

$$\phi_T(B) = (1 - B)^2$$

and

$$\phi_S(B) = (1 - \alpha B)(1 + B + \ldots + B^{11})$$

Figure 3.1 CON-BPNE1



and where U_t, V_t and I_t are mutually uncorrelated white noise series whose known variances are given by σ_u^2, σ_v^2 and 1, respectively.

Given initial values for T_t and S_t and the covariances of these initial values, the Kalman smoother can be used to obtain the least squares estimates of T_t, S_t for $1 \le t \le N$, given $y_1, \ldots, y_N$ (see Brotherton and Gersch 1981). When $\alpha = 1$ and the initial covariances are set equal to zero, the values of S_t and T_t so obtained are those produced by Akaike's procedure for given values of d^2 and RIGID, with z equal to zero, i.e., with the constraint expressions

$$(S_i + S_{t-1} + \ldots + S_{i-11})^2$$

dropped from his model (and with ORDER=2 and SORDER=1). The same calculation with α equal to zero yields the values obtained by dropping instead the constraint expressions $(S_i - S_{i-12})^2$. (Obvious modifications of (4.2) and (4.3) in this procedure yield Akaike's values for different choices of ORDER and SORDER.)

With both seasonal constraint terms present, it is more difficult to exactly relate Akaike's procedure to ARIMA models. George Tiao has obtained a connection with the models defined by (4.2) and (4.3), which we wish to discuss briefly.

For known values of α, σ_u^2 and σ_v^2, the estimates of $\hat{T}_{sx,t}$ and $\hat{S}_{sx,t}$, of T_t and S_t, obtained by formally applying the least squares signal extraction procedures from the theory of *stationary* time series (Koopmans 1974, p. 148) are solutions of

$$R(B)\hat{T}_t = \phi_S(B^{-1})\phi_S(B)\sigma_u^2 y_t \qquad (4.4)$$

and

$$R(B)\hat{S}_t = \phi_T(B^{-1})\phi_T(B)\sigma_v^2 y_t \qquad (4.5)$$

respectively, where the "shifted polynomial" $R(B)$ is given by

$$R(B) = \sigma_u^2\phi_T(B^{-1})\phi_T(B) + \sigma_v^2\phi_S(B^{-1})\phi_S(B) + \phi_T(B^{-1})\phi_T(B)\phi_S(B^{-1})\phi_S(B)$$

In a study note attached as an appendix to these comments, George Tiao shows that the estimates $\hat{T}_{a,t}$ and $\hat{S}_{a,t}$ of T_t and S_t obtained from Akaike's procedure with ORDER=2 and SORDER=1 must also satisfy (4.4) and (4.5), with $\alpha = \{(z^2 + 2) - ((z^2 + 2)^2 - 4)^{1/2}\}/2$ and

$$\sigma_u^2 = s^{-2}d^{-2}, \sigma_v^2 = \alpha d^{-2}$$

for values of t in the range

$$\deg \phi_T \cdot \phi_S \le t \le N - \deg \phi_T \cdot \phi_S \qquad (4.6)$$

where N denotes the length of the span of the observed series y_t being adjusted and $deg\ \phi_t \cdot \phi_S$ denotes the degree of the product polynomial $\phi_T(B)\phi_S(B)$. It follows that for these t values, the differences

$$\Delta\hat{T}_t = \hat{T}_{sx,t} - \hat{T}_{a,t}$$

must satisfy

$$R(B)\ \Delta\hat{T}_t = 0 \qquad (4.7)$$

and similarly for $\Delta\hat{S}_t = \hat{S}_{sx,t} - \hat{S}_{a,t}$.

These conditions are not terribly restrictive: The solutions of (4.7) need not converge to zero as t becomes large and, when they do converge to zero, the geometric rate of convergence can be slow. (See the comments regarding the roots of $R(B)$ given below.) Further, when $N = 48$, as Akaike recommends, and when deg $\phi_t \cdot \phi_S = 14$, as it is in this example, the 28th-order difference equation (4.7) is required to hold for only twenty t values, $14 \leq t \leq 34$.

Along with these qualifying remarks, it should also be mentioned, however, that the equations (4.2) and (4.3) do correctly suggest two important features of the trends and seasonals produced by Akaike's method, namely, the sensitivity to starting values mentioned by Akaike (1980) and the kind of forecasts of T_t and S_t obtained when, as Akaike mentions as a possibility, the y_t corresponding to the t values for which forecasts are desired are declared to be missing values. In several examples we looked at, the T_t forecasts looked like straight-line extrapolations from the last few T_t in the range of observation, and the S_t forecasts were very close to the periodic extension of the last twelve S_t values associated with the observed y_t.

It is easy to generalize Tiao's argument to cover other values of ORDER and SORDER. When ORDER=1, one obtains that $\phi_T(B) = 1-B$. When SORDER=2, our attempts to construct $\phi_S(B)$ (which has degree 24) by finding the roots of $\phi_S(B^{-1})\phi_S(B)$ met with numerical difficulties. The results obtained suggest that $\phi_S(B)$ has double roots very close to $\exp(\pm ik\pi/6)$, k=2,3,4,5, along with single roots close to -1 and $\exp(\pm \pi/6)$, and no other roots very close to the unit circle. In all of the cases we considered, for SORDER=1,2, all but a few of the roots of $R(B)$ were close to (but not on) the unit circle, the closest roots being near to $\exp(\pm ik\pi/6)$, k=0,1,2,3,4,5,6.

For example, with ORDER=2, SORDER=1, and d^2=19, the roots of $R(B)$ are the seven numbers given below, along with their reciprocals, complex conjugates, and conjugate reciprocals.

Table 4.1 BASIC ROOTS $re^{i\theta}$ of $R(B)$

r	.71	.75	.98	.98	.98	.98	.99
$2\pi/\theta$	18.7	∞	2.4	3	4	6	12

5. CONCLUDING REMARKS

The number of examples we have considered is too small to support firm conclusions. Coupled with the results of Akaike (1980) and the present paper by Akaike and Ishiguro, however, our results would seem to offer additional evidence that BAYSEA may find a place among the handful of seasonal adjustment methods which enjoy or can expect to enjoy wide usage. Given the ease with which calendar-effect variables, intervention variables, and other special trend, or seasonal or irregular compensatory adjustments can be incorporated into BAYSEA, this would be welcome.

I would like to thank Ted Holden for computer assistance in the preparation of these comments.

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APPENDIX

STUDY NOTES BY G. C. TIAO ON AKAIKE'S SEASONAL ADJUSTMENT PROCEDURE

Here we shall use t for i and the backshift operator B such that $By_t = y_{t-1}$. Thus, let $y_t = T_t + S_t + I_t$ where T_t, S_t and I_t are unobservable components. We show the relationships between Akaike's (1980) Bayesian minimization procedure

$$f = \sum_{t=1}^{N}(y_t - T_t - S_t)^2 + d^2\{s^2(T_t - 2T_{t-1} + T_{t-2})^2 + (S_t - S_{t-12})^2 + z^2(S_t + \ldots + S_{t-11})^2\} \qquad (A.1)$$

and the model-based procedures advocated in papers by Box, Cleveland, Hillmer, Pierce and Tiao.

From (A.1),

$$\frac{1}{2}\frac{\partial f}{\partial T_t} = (Y_t - T_t - S_t)$$
$$+ d^2s^2\{6T_t - 4(T_{t+1} + T_{t-1})$$
$$+ (T_{t+2} + T_{t-2})\} = 0$$
$$\frac{1}{2}\frac{\partial f}{\partial S_t} = (Y_t - T_t - S_t)$$
$$+ d^2\{(2S_t - S_{t-12} - S_{t+12})$$
$$+ z^2[12S_t + 11(S_{t+1} + S_{t-1})$$
$$+ 10(S_{t+2} + S_{t-2}) + \ldots$$
$$+ (S_{t+11} + S_{t-11})]\} = 0$$

Thus, we have that

$$Q_1T_t = Y_t - S_t$$
$$Q_2S_t = Y_t - T_t \qquad \text{(A.2)}$$

where

$$Q_1 = 1 + d^2S_2(1 - B)^2(1 - F)^2$$
$$Q_2 = 1 + d^2[(1 - B^{12}) + z^2 \cup(B)\cup(F)]$$
$$= 1 + d^2\cup(B)\cup(F)[z^2 + (1 - B)(1 - F)]$$
$$= 1 + c^2\cup(B)\cup(F)(1 - \alpha B)(1 - \alpha F)$$

and where $F = B^{-1}$, $\cup(B) = 1 + B + \ldots + B^{11}$, $\alpha z^2 = (1 - \alpha)^2$ and $c^2 = \alpha^{-1}d^2$. It follows that for large N and t not close to the end points

$$\hat{T}_t = (Q_1Q_2 - 1)^{-1}(Q_2 - 1)Y_t \qquad \text{(A.3)}$$
$$\hat{S}_t = (Q_1Q_2 - 1)^{-1}(Q_1 - 1)Y_t$$

Now, let us suppose that T_t and S_t follow the ARMA model

$$\phi_T(B)T_t = \theta_T(B)U_t$$
$$\phi_S(B)S_t = \theta_S(B)V_t \qquad \text{(A.4)}$$

where $\{U_t\}$, $\{V_t\}$ and $\{I_t\}$ are uncorrelated white noise processes with variances σ_u^2, σ_v^2 and 1, respectively. Then, from signal extraction theory, the best estimates of T_t and S_t are

$$\hat{T}_t = R^{-1}\phi_S(B)\phi_S(F)\theta_T(B)\theta_T(F)\sigma_u^2y_t$$
$$\hat{S}_t = R^{-1}\phi_T(B)\phi_T(F)\theta_S(B)\theta_S(F)\sigma_v^2y_t \qquad \text{(A.5)}$$

where

$$R = \sigma_u^2\phi_S(B)\phi_S(F)\theta_T(B)\theta_T(F)$$
$$+ \sigma_v^2\phi_T(B)\phi_T(F)\theta_S(B)\theta_S(F)$$
$$+ \phi_T(B)\phi_T(F)\phi_S(B)\phi_S(F)$$

By comparing (A.3) with (A.5), we can simply take $\phi_S(B) = (1 - \alpha B)\cup(B)$, $\phi_T(B) = (1 - B)^2$, $\theta_T(B) = \theta_S(B) = 1$, $\sigma_v^2 = c^{-2}$ and $\sigma_u^2 = s^{-2}d^{-2}$. That is, the models for T_t and S_t are, respectively,

$$(1 - B)^2T_t = U_t, \qquad \sigma_u^2 = s^{-2}d^{-2}$$
$$(1 - \alpha B)(1 + B + \ldots + B^{11})S_t = V_t, \sigma_v^2 = c^{-2}$$
$$\text{(A.6)}$$

To partially check the appropriateness of (A.6) in practice, note that models in (A.6) imply that the overall model for Y_t is

$$(1 - \alpha B)(1 - B)(1 - B^{12})Y_t = \theta(B)a_t \qquad \text{(A.7)}$$

where $\sigma_a^2\ \theta(B)\ \theta(F) = R$. Thus, we would argue that the minimization procedure in (A.1) would be consistent with information from the data if the overall model of Y_t is of the form (A.7). Note from (A.2) that $(1 + \alpha^2)\alpha^{-1} = (z^2 + 2)$ so that from Akaike's paper, the value of α is fixed. If, on the other hand, the model for Y_t is vastly different from (A.7), then the use of (A.1) would be thrown in doubt.

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COMMENTS ON "COMPARATIVE STUDY OF THE X−11 AND BAYSEA PROCEDURES OF SEASONAL ADJUSTMENT" BY HIROTUGU AKAIKE AND MAKIO ISHIGURO

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In this interesting paper, the Census Bureau X−11 method is compared with the BAYSEA seasonal adjustment method developed by Akaike. The BAYSEA procedure assumes that the observed series Y_t (or the log of Y_t) can be decomposed into three additive unobserved components: The trend-cycle component TC_t, the seasonal component S_t, and the irregular component I_t. When estimating the three components the objective function applied is

$$\min\{Y_t - TC_t - S_t)^2 + d^2[s^2(\Delta_1^\delta TC_t)^2 + (\Delta_{12}^\lambda S_t)^2 + z^2(\sum_{j=0}^{11} S_{t-j})^2]\}$$

where s, z, δ, and λ are constants chosen by the user. Δ_i is the difference operator $(1 - L)^i$, L being the lag operator, while a grid search is used to estimate d. The rationale behind the term $(\Delta_1^\delta TC_t)$ is that the trend-cycle component ought to be smooth, while the rationale behind the term $(\Delta_{12}^\lambda S_t)$ is that the seasonal component should be stable. The term $(\sum_{j=0}^{11} S_{t-j})$ is included in order to keep the 12-month sum for the seasonal component close to zero.

Obviously, several reservations can be made against the arbitrary criteria that the trend-cycle component should be smooth and the seasonal component stable. (See Hylleberg, 1981, ch 2.) But when applying the BAYSEA procedure, it becomes important to realize that one of the main objectives of deseasonalizing single economic time series is to promote the prediction of turning points. As a consequence, values of δ less than or equal to 2 should be avoided as they imply biasing the estimated trend-cycle component towards being constant ($\delta = 1$) or linear ($\delta = 2$).

Nonetheless, in applying the BAYSEA procedure to actual economic time series, Akaike and Ishiguro search over values of δ equal to 1 and 2 only.

Of course, these biases will be small if sufficiently low values of s are applied, but it seems preferable to apply higher degree smoothness priors, i.e., higher values of δ and correspondingly higher values of s.

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RESPONSE TO DISCUSSANTS

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We are grateful that the discussants took up several important points that we did not discuss in our paper. Before answering the points, we would like to emphasize the conventional character of the BAYSEA procedure. We consider it to be a simple flexible rule that will allow a reasonable measurement of the trend and seasonal components. Thus, we do not think it sufficiently useful for forecasting. For that purpose, more detailed analysis and modeling of the behavior of these components are necessary. The output of BAYSEA will provide a starting point for this type of research.

As to the first point of Dr. Dagum, the frequency response analysis of BAYSEA, we must emphasize the data-adaptive nature of BAYSEA. The frequency response analysis is a procedure developed mainly for the analysis of a constant linear system. In the BAYSEA procedure, not only the parameter *d* but also other parameters, such as ORDER, SORDER and RIGID, are chosen adaptively. Even when these latter parameters are fixed, the adaptive choice of *d* gives the procedure an essentially nonlinear characteristic. For the analysis of such a system, it is more informative to observe the responses of the system under typical operating conditions, as is done in the test of audio amplifiers. Figures 1 and 2 show the responses of X−11 and BAYSEA (2, 1, 1.0), without extreme value corrections, to the square wave input. The generation of the spurious seasonal component by X−11 is much more significant than that of BAYSEA. Figures 3 and 4 show similar results obtained by X−11 and BAYSEA (1, 1, 1.0) for a white noise input. The spurious movement of the trend component generated by X−11 clearly demonstrates the undesirably high sensitivity of the trend-cycle filter to the irregular component. These simple examples amply explain the reason why we restricted our analysis to the empirical comparison of the 14 sets of data provided by the Bureau of the Census. These data are full of complexities that cannot be easily simulated by simple artificial inputs. It is common knowledge that the frequency response characteristic is only supplementary information in testing an audio equipment of high quality. Here, judgements by experts are still playing a dominant role, i.e., we are still in the process of searching for a decisive characteristic in choosing a system. Maybe this is an unending process. We believe that the situation is much the same with seasonal adjustment and we pay very much attention to the opinions of experts in the area of practical application.

We are glad to see that Dr. Dagum is in agreement with us and consider that the amount of revision itself is not a decisive characteristic for the comparison of seasonal adjustment procedures. We believe that our numerical results have shown that BAYSEA produced smaller revisions compared with X−11, when the final outputs were similar. BAYSEA often produced more reasonable trend estimates, as is shown by the results of figures 6 and 7, without undue increase of the amount of revision. At least, we can say that BAYSEA is a procedure that produces results essentially different from those by X−11, with respect to some characteristics that are perhaps of practical importance.

Coming to the last point, the operational state of BAYSEA, we must mention that the program made available to the discussants was a prototype designed for the ease of understanding and modification of the procedure by the user. However, even with this program, the CPU time to produce the result of figure 1 by our computer was 18.66 seconds and included the computation of covariance sequences and spectra of the component series. The X−11 ARIMA took 4.76 seconds for the same data, without covariance and spectrum computation. We already have a faster version of BAYSEA.[1] By use of this version, the CPU time for the adjustment of the same data was 4.57 seconds, with slightly reduced output of spectra. Thus, we can see that the computational efficiency is not a main problem. Further improvement of computational efficiency of BAYSEA is not quite improbable. Due to the simplicity of the structure, anyone who is interested in the procedure can easily develop his or her own version of BAYSEA.

The printout of the necessity of trying different ORDER or SORDER shows that the value of the parameter *d* is hitting the boundary of its possible values and is usually very useful in the search for an appropriate model. However, the final decision should depend on ABIC's, as described in our paper.

Dr. Findley supplemented our analysis by checking the performance of BAYSEA with respect to the transformation of data, adjustment of short series, and the analysis of synthesized series. We are glad to see that his findings are consistent with our experience. The comparison with X−11 based on average absolute percentage changes will be of particular interest to those who are familiar with the use of this statistic. Certainly, we are pleased to see that in Dr. Findley's experiment the BAYSEA procedure produced results better than, or at least equivalent to, those by X−11.

The analysis through time series modeling developed by Professor George Tiao is very valuable. As we already mentioned, our procedure is very data adaptive and, in that sense, responsive to some structural change of the generating

[1]Ishiguro, M. 1981, "Computationally Efficient Implementation of a Bayesian Seasonal Adjustment Procedure," Research Memo. 24, March 1981, The Institute of Statistical Mathematics, Tokyo, Japan.

mechanism of the original series. As is noticed by Dr. Findley, this characteristic may not be adequately described by stationary time series modeling. Nevertheless, this type of analysis might be helpful when our procedure encounters some particular type of difficulty.

Dr. W. S. Cleveland's suggestion of robustification of BAYSEA against outliers seems very natural and interesting. The tentative procedure of outlier correction in the present version of BAYSEA is based on a Bayesian modeling. Although it works fairly well, it is extremely time consuming. We hope to investigate the possibility of implementing a computation by more efficient procedures of outlier correction.

Figure 1. RESPONSE OF X-11 TO THE SQUARE WAVE TEST

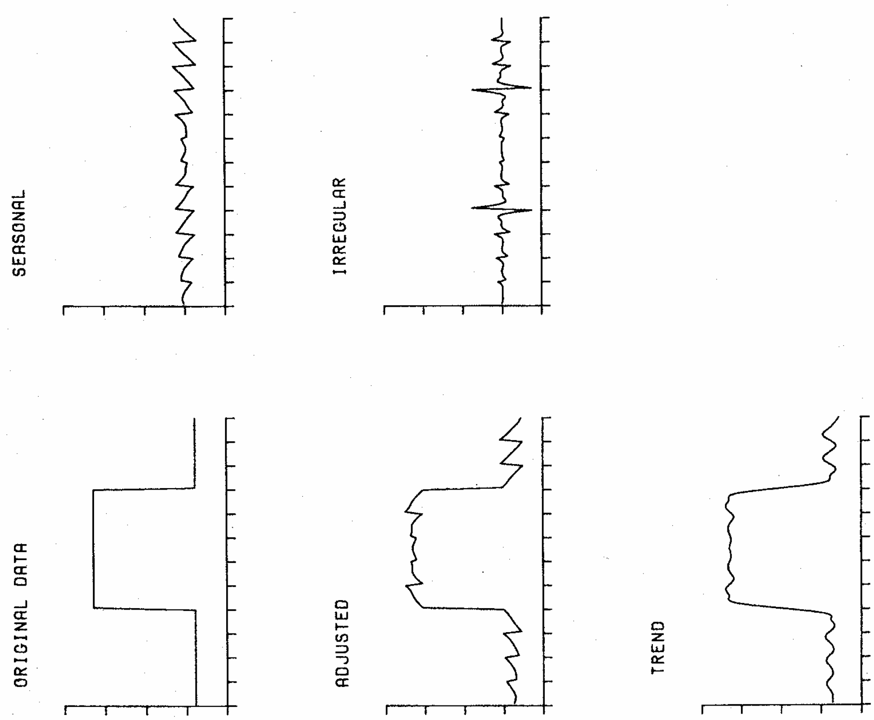


Figure 2. RESPONSE OF BAYSEA TO THE SQUARE WAVE TEST

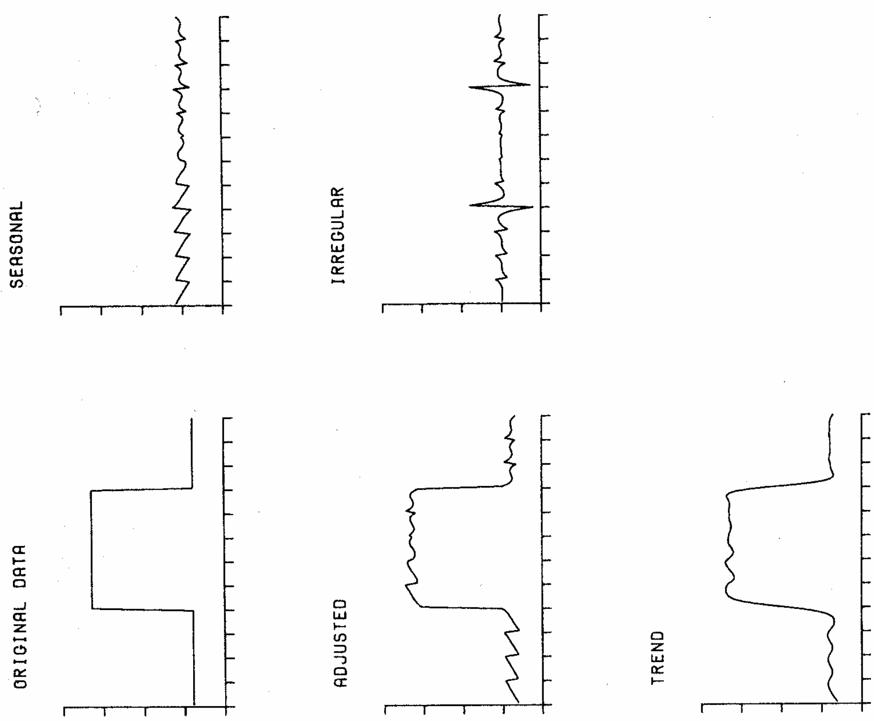


Figure 3. RESPONSE OF X-11 TO THE WHITE NOISE TEST

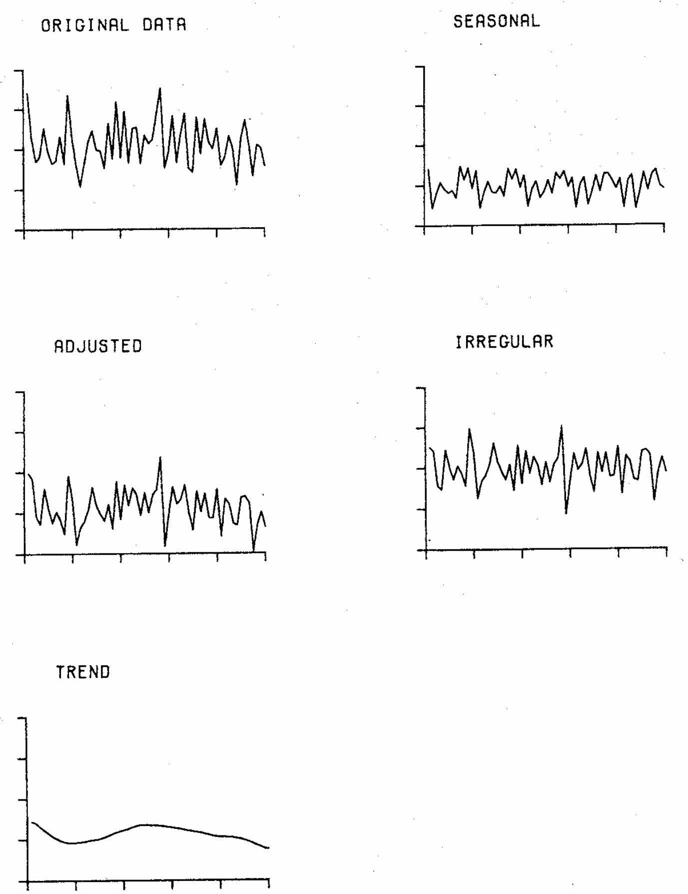
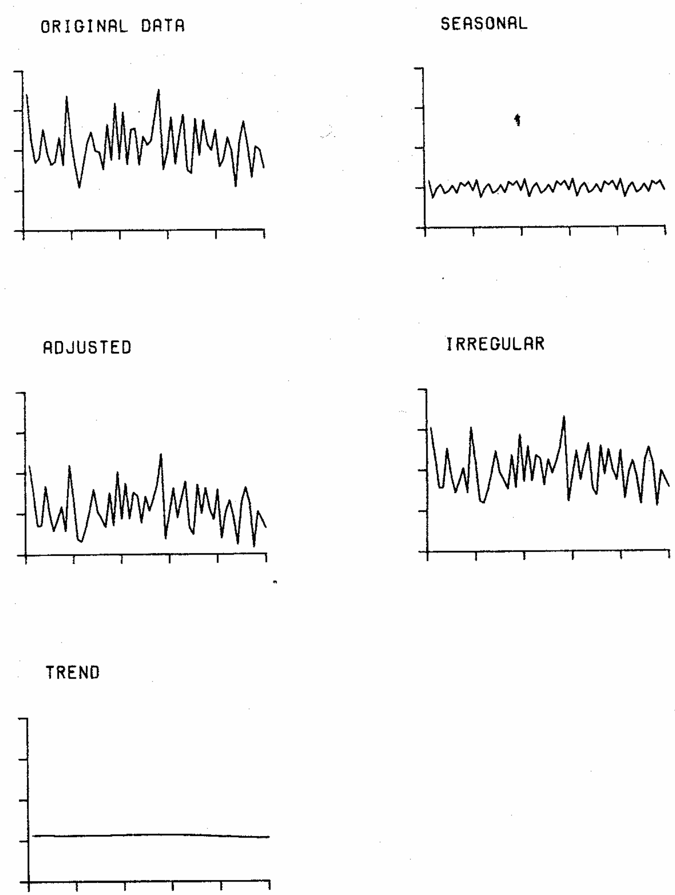


Figure 4. RESPONSE OF BAYSEA TO THE SQUARE WAVE TEST



COMMENTS ON "COMPARATIVE STUDY OF THE X-11 AND BAYSEA PROCEDURES OF SEASONAL ADJUSTMENT" BY H. AKAIKE AND M. ISHIGURO

Adi Raveh

Akaike and Ishiguro formulated the decomposition of a given time series into trend, seasonal, and irregular components as a minimization problem. They assume the additive structure $Y_i = T_i + S_i + I_i$ where Y_i denotes the observation at time i and T_i, S_i, and I_i denote the trend, seasonal and irregular component, respectively.

They expect the two systematic components, trend and seasonal, to have at least locally smooth behavior in linear terms and a stable yearly pattern, respectively. This expectation is represented by requiring that

$$[S^2(T_i - 2T_{i-1} + T_{i-2})^2 + (S_i - S_{i-12})^2 + Z^2(S_i + S_{i-1} + \ldots + S_{i-11})^2]$$

be small. The last term within the brackets is added to keep the 12 months sum of the seasonal variations close to zero. Here S and Z are properly chosen constants. The authors also expect that the systematic part $T_i + S_i$ will not deviate significantly from the original observation Y_i. This suggests the minimization of

$$[Y_i - T_i - S_i]^2 \tag{1}$$

Consideration of the above two quantities leads to the minimization of

$$[Y_i - T_i - S_i]^2 + d^2[S^2(T_i - 2T_{i-1} + T_{i-2})^2 + (S_i - S_{i-12})^2 + Z^2(S_i + \ldots + S_{i-11})^2] \tag{2}$$

where d is a properly chosen constant. Estimates of the two systematic components, T_i and S_i can be obtained by minimizing (2) using the standard procedure of least squares. Of course, the three parameters (weights) d^2, S^2 and Z_2 must be specified before the procedure becomes operational. This problem is solved as a problem of statistical model selection using AIC. The very same procedure was suggested recently by Schlicht (1981). His method minimizes the equation (2) for monthly series. His α, β, and γ play the same role as d^2S^2, d^2, and d^2Z^2 in Akaike's and Ishiguro's paper.

Schlicht has derived a unique solution to the minimization problem. He claims that the appropriate choice of parameter values will depend on the shape of the time series, but he did not show how to choose the desired values of the parameters. Thus, if the seasonal pattern is nearly fixed over the whole period of observation, a very high β (not α as is mistakenly claimed by Schlicht) would be appropriate. High values of γ are required to keep the seasonally adjusted data (SAD) in the same scale as the original data. In other words, the sum of the original series and the SAD for any 12 consecutive observations will be about the same. Procedures to check that the pattern is constant and to choose the parameter values are not given by Schlicht.

Both methods trade off among the four components of the overall loss function by means of the three weight parameters. Thus, for example, when β increases to infinity, the moving seasonality becomes fixed. When $\beta = 0$, the moving seasonality behaves like an irregular component with every consecutive 12 values adding up to 12 approximately, depending on the value of γ.

Both procedures try to estimate as smooth a trend as possible in terms of local linearity. The part of the loss function that relates to trend is based on squares of the second differences and is provided by minimizing

$$LIN(T) = \sum_{i=3}^{N}(\Delta T_i - \Delta T_{i-1})^2 \geqslant 0 \tag{3}$$

where $\Delta T_i = T_i - T_{i-1}$. $LIN(T) = 0$ if and only if the trend is perfectly linear. $LIN(T) \doteq 0$ (relatively, close to zero), if the trend is locally linear, namely, if there are very few turning points and between turning points the trend is linear.

Linearity conditions for the trend are: $T_i - T_{i-1} = T_{i-1} - T_{i-2}$ for all $i = 3, \ldots, N$, or $\Delta T_i = \Delta T_{i-1}$, or $\Delta^2 T_i = 0$. Another point of view which is slightly different is that the trend will be as smooth as possible in terms of local *monotonicity*. The conditions for weak monotonicity are

$$(T_i - T_{i-1})(T_{i-1} - T_{i-2}) \geqslant 0 \text{ for all } i = 3, \ldots, N$$

or

$$\Delta T_i \Delta T_{i-1} \geqslant 0$$

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or

$$\Delta T_i \Delta T_{i-1} = |\Delta T_i \Delta T_{i-1}| \text{ for all } i = 3, \ldots, N$$

Thus, smooth trend can be measured by a *coefficient of local monotonicity* given in (4) below.

$$MON(T) = \mu_\Delta = \frac{\sum_{i=3}^{N} \Delta T_i \cdot \Delta T_{i-1}}{\sum_{i=3}^{N} |\Delta T_i \cdot \Delta T_{i-1}|} \quad (4)$$

$\mu_\Delta = 1$ if and only if the trend is perfectly weak monotone with either positive or negative slope. If very few turning points exist (relative to the length of the series) and between them the trend is monotone then $\mu_\Delta \doteq 1$, and we call it local monotonicity. The least monotone trend would be obtained for series that their slope change rapidly. Thus for the following series: $a,b,a,b, \ldots, b$ where $a \neq b$, the coefficient $\mu_\Delta = -1$.

In terms of monotonicity, a smooth trend is achieved as μ_Δ is increased and thus max μ_Δ is desired. To combine this loss function with the overall loss function, let us consider the minimization of the quantity:

$$[Y_i - T_i - S_i]^2 + A\,|[\Delta T_i \Delta T_{i-1} - |\Delta T_i \Delta T_{i-1}]|^P + B(S_i - S_{i-12})^2 + C(S_i + \ldots + S_{i-11})^2 \quad (5)$$

where A, B, C, P are properly chosen constants (usually P = 1 or 2 can be used). The solution can be obtained by numerical algorithms for minimization such as in Zangwill (1967). It is obvious that $LIN(T) = 0$ yields $MON(T) = 1$, but not vice versa. In other words, a trend can be very smooth in monotone terms but at the same time may be far from being a linear one, as in figure 1, below. By adopting this minimization formula, a smoother trend should be obtained in terms of local monotonicity. By minimizing (2) for a discontinuous and polytone (namely, local monotone) trend, one could obtain a local linear trend as an estimate although such does not exist for the data. In figure 2, original series and trend estimation are exhibited for wholesale inventories of grocery stores in the United States for January 1967 to July 1980. In 2a and 2b, estimates of global monotone (nearly linear) trend and local monotone (polytone of order 3) trend are obtained, respectively. It seems that the estimation in 2b represents the intrinsic (hidden) local trend more accurately, especially around the turning points.

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Figure 1. A LOCAL POLYTONICITY SERIES WHICH IS NOT SMOOTH IN THE LOCAL LINEARITY SENSE

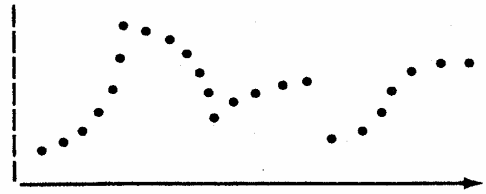
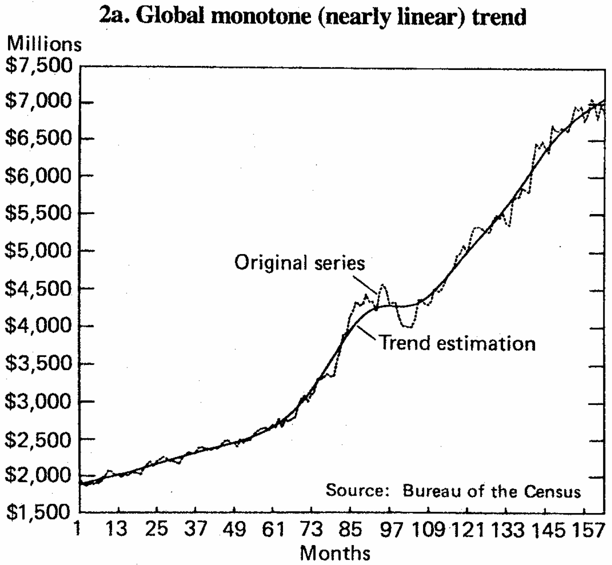
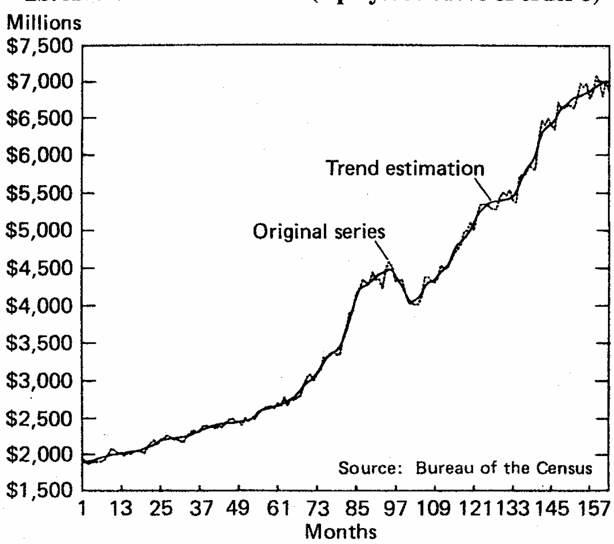


Figure 2. WHOLESALE INVENTORIES OF GROCERY STORES: JANUARY 1967 TO JULY 1980

2a. Global monotone (nearly linear) trend



2b. A local monotone trend (a polytone curve of order 3)



REPLY TO DR. RAVEH'S COMMENT

H. Akaike and M. Ishiguro

We are grateful to Dr. Raveh for pointing out the distinction between the Schlicht's constrained least squares and our Bayesian approach. As was mentioned in Dr. Dagum's discussion, the constrained least squares approach has a long history. It is our use of ABIC, or the likelihood of the Bayesian model, that made BAYSEA a viable alternative to the X−11 procedure.

The concept of local monotonicity is interesting. However, the minimization of (5) will allow irregular variations of the trend in monotone phases and only curb the behavior at turning points. This seems to be somewhat in contradiction to Dr. Raveh's final statement. It may also be mentioned that the application of the Bayesian modeling to (5) may not be quite feasible as a practical procedure.