MODELING CONSIDERATIONS IN THE SEASONAL ADJUSTMENT OF ECONOMIC TIME SERIES

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1. INTRODUCTION

Work leading to currently used methods of seasonal adjustment began with the link-relative method of Persons (1919). His efforts motivated others in the 1920's and 1930's to consider the problem of seasonal adjustment. Macauley's (1931) development of the ratio-to-moving-average method was particularly important because it is the basis for the current Census X−11 method. Macauley and others borrowed the tool of moving averages from actuaries to smooth their data, rather than fit explicit functions to the data, because they felt that the trend and cycle components varied smoothly over time but they were, ". . . not necessarily representable throughout [their] length by any simple mathematical equation" (Macauley 1931, p. 32).

In 1954, Julius Shiskin started doing seasonal adjustment on electronic computers at the U.S. Census Bureau. This permitted the use of methods which involved elaborate calculations and led to experimentation with the ratio-to-moving-average method. Improvements were made in the Census adjustment methods over the next 10 years, culminating with Census Method II, X−11 Variant (Shiskin, Young, and Musgrave 1967) which was adopted at the Census Bureau in 1965 and is today widely used. The X−11 procedure was developed over a number of years from empirical experimentation, rather than from theoretical considerations. In this paper, we refer to X−11 and other seasonal adjustment procedures that have evolved from similar considerations as *empirical adjustment* methods. Since 1965, there have been other empirical adjustment methods developed, including X−11 ARIMA (Dagum 1975) and Bell Lab's SABL (Cleveland, Dunn, and Terpenning 1978).

Signal extraction. Seasonal adjustment can be viewed as estimation of an (unobserved) nonseasonal component, N_t, or as estimation and removal of an (unobserved) seasonal component, S_t, from an observed time series Z_t. If the decomposition can be viewed as additive, $Z_t = S_t + N_t$, or additive for some suitable transformation of Z_t (such as ln Z_t, in which case $Z_t = e^{S_t} \cdot e^{N_t}$), then it is natural to use signal extraction theory to do seasonal adjustment. The signal extraction problem was solved for stationary time series by Kolmogorov and Wiener in the 1940's; their results were extended to nonstationary series by Hannan (1967), Sobel (1967), and Cleveland and Tiao (1976). These results assume that the models for S_t and N_t are known, but this assumption is unrealistic for economic time series. We can model the observed data, Z_t, and this model will provide some information about the stochastic structure of S_t and N_t; beyond this, models for S_t and N_t have to be assumed in order to apply signal extraction theory. This is not a weakness of the signal extraction approach to seasonal adjustment; rather, it emphasizes the arbitrariness inherent in any seasonal adjustment procedure, *including* empirical methods.

Model-based procedures. In recent years, a number of seasonal adjustment methods have been proposed which use signal extraction theory with explicit statistical models for S_t and N_t (e.g., Box, Hillmer, and Tiao 1978, Pierce 1978, Burman 1980, and Hillmer and Tiao 1982). We call such procedures *model-based* methods. The publication of the book by Box and Jenkins (1970) was an important factor in the development of model-based approaches because it described relatively simple techniques for modeling seasonal time series. The work of Cleveland and Tiao (1976) represents an early attempt to link empirical methods with a model-based procedure. The recent research efforts into model-based methods are further attempts to integrate desirable features of empirical methods with statistical theory for the purposes of better understanding the problem of seasonal adjustment and of making improvements in procedures.

While we believe that model-based adjustment methods can offer improvements over existing empirical approaches, such as X−11, this is not the whole story. The X−11 procedure

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This research was supported in part by the U.S. Army Research Office under Grant No. DAAG 29−78−G−1066, by the Bureau of the Census under JSA 80−10 and JSA 81−2, and by the University of Kansas General Research Allocation 3438−2038. Part of the research was completed while the first two authors were participants in the ASA/Census fellowship program, which was supported by grants from the Census Bureau and the National Science Foundation.

also deals with other important problems, including trading-day and holiday variation and the treatment of outliers. Any approach to seasonal adjustment must ultimately deal with these problems. We think it sensible that all these issues be addressed at the modeling stage.

Objectives of this paper. The primary objectives of this paper are (i) to present a comprehensive model-based approach to seasonal adjustment and (ii) to compare and contrast this approach with X−11 and related empirical methods. Section 2 gives a summary of the statistical theory underlying the model-based approach. Section 3 discusses modeling of series with trading-day and holiday variation, and section 4 deals with the problem of outliers. In section 5, we provide some theoretical and empirical comparisons of the X−11 and the X−11 ARIMA procedures with our proposed procedure in terms of revisions. Finally, section 6 discusses some issues in seasonal adjustment as they relate to model-based and empirical methods.

2. AN ARIMA MODEL-BASED APPROACH TO SEASONAL ADJUSTMENT

In this section, we summarize the theory behind the ARIMA model-based seasonal adjustment method developed in Hillmer and Tiao (1982). A similar treatment can be found in Burman (1980). We assume that an observable time series or some appropriate power transformation of the series, Z_t, can be represented as

$$Z_t = S_t + N_t \tag{2.1}$$

where S_t and N_t are mutually independent seasonal and nonseasonal components. If desired, N_t can be further decomposed into trend and noise components; however, because the usual practice in the United States is to publish estimates of N_t, we shall only consider decomposition into two components.

If in (2.1) the stochastic structures of S_t and N_t are known, then minimum mean squared error estimates of S_t and N_t using the observed values of Z_t are readily obtained from the theory of signal extraction given in Whittle (1963) and Cleveland and Tiao (1976). In particular, suppose the models for the components are

$$\phi_s(B)S_t = \eta_s(B)b_t \tag{2.2}$$
$$\phi_N(B)N_t = \eta_N(B)c_t$$

where the pairs of polynomials in the backshift operator B, $\{\phi_s(B), \eta_s(B)\}$, $\{\phi_N(B), \eta_N(B)\}$, and $\{\phi_s(B), \phi_N(B)\}$ have no common zeros, and b_t and c_t are mutually independent, i.i.d. $N(0, \sigma_b^2)$ and $N(0, \sigma_c^2)$, respectively. It follows from (2.1) that the model for Z_t is

$$\phi(B)Z_t = \theta(B)a_t \tag{2.3}$$

where (Cleveland 1972) $\phi(B) = \phi_s(B)\phi_N(B)$, and $\theta(B)$ and σ_a^2 are determined from the equation

$$\sigma_a^2\frac{\theta(B)\theta(F)}{\phi(B)\phi(F)} = \sigma_b^2\frac{\eta_s(B)\eta_s(F)}{\phi_s(B)\phi_s(F)} + \sigma_c^2\frac{\eta_N(B)\eta_N(F)}{\phi_N(B)\phi_N(F)} \tag{2.4}$$

with $F = B^{-1}$. Furthermore, when all the zeros of $\phi_s(B)$ and $\phi_N(B)$ are on or outside the unit circle, the estimated seasonal and nonseasonal components are

$$\hat{S}_t = W_s(B)Z_t \text{ and } \hat{N}_t = W_N(B)Z_t \tag{2.5}$$

where

$$W_s(B) = \frac{\sigma_b^2\phi(B)\phi(F)\eta_s(B)\eta_s(F)}{\sigma_a^2\theta(B)\theta(F)\phi_s(B)\phi_s(F)} = \frac{\sigma_b^2\eta_s(B)\eta_s(F)}{\sigma_a^2\theta(B)\theta(F)}\phi_N(B)\phi_N(F)$$

and

$$W_N(B) = \frac{\sigma_c^2\phi(B)\phi(F)\eta_N(B)\eta_N(F)}{\sigma_a^2\theta(B)\theta(F)\phi_N(B)\phi_N(F)} = \frac{\sigma_c^2\eta_N(B)\eta_N(F)}{\sigma_a^2\theta(B)\theta(F)}\phi_s(B)\phi_s(F)$$

In practice, the S_t and N_t series are unobservable. Thus, without additional information, the component models (2.2) are unknown so that the weight functions $W_s(B)$ and $W_N(B)$ and, therefore, the estimates $\hat{S}_t$ and $\hat{N}_t$ cannot be computed. However, an accurate estimate of the model (2.3) can be obtained from the observable Z_t series. Consequently, it is of interest to investigate to what extent the component models can be determined from the model for Z_t, and to what extent prior knowledge about the component models is required to achieve a decomposition.

2.1. Prior Knowledge About S_t

Because X−11 has been widely used for many years, some of the characteristics it attributes to the seasonal component must have been regarded as desirable by users of seasonally adjusted data. In particular, additive X−11 produces estimates of S_t which aproximately repeat every year and approximately sum to zero over any 12 consecutive months. Also, Young (1968) shows that the X−11 procedure for estimating S_t may be approximated by a linear filter with weights that decrease from the center toward the ends of the filter. This feature implicitly assumes that S_t evolves over time. Judging from these considerations, it seems reasonable that S_t should be a (nondeterministic) stochastic process, but that locally a regular seasonal pattern should be preserved.

If it is sensible to require that the sum of S_t over any 12 consecutive months should vary about zero, then the moving sum $U(B)S_t$, where $U(B) = 1 + B + \ldots + B^{11}$, should be a stationary time series with mean zero. In this case, we can write $U(B)S_t = \eta_s(B)b_t$, where the b_t's are i.i.d. $N(0, \sigma_b^2)$ and $\eta_s(B)$ is as yet of unspecified degree. Notice that

$E[U(B)S_t] = E[\eta_s(B)b_t] = 0$. If we also require S_t to locally follow a fixed seasonal pattern, the forecasting function at a given time origin of the model for S_t should follow a fixed pattern of period 12 and should sum to zero over 12 consecutive months. In other words, the model for S_t should not allow for predictable changes in the seasonal pattern; such changes should be part of the trend component. It is easy to show that these requirements are equivalent to restricting $\eta_s(B)$ to be of degree at most 11. Thus, we are led to the following model for the stochastic seasonal component S_t

$$U(B)S_t = \eta_s(B)b_t \tag{2.6}$$

where $\eta_s(B)$ is a polynomial in B of degree at most 11. Seasonal components following (2.6) will locally follow a fixed pattern, but as long as $\sigma_b^2 > 0$ the forecasting function of (2.6) will be continually updated as the time origin advances, thus the pattern in S_t will evolve over time.

2.2. Restrictions Imposed by the Data

We shall investigate how information available from the data embodied in a known model for Z_t, together with the additivity and independence assumptions (2.1), restrict the possible models for S_t and N_t. From the discussion in the previous section, Z_t will have a seasonal component if $\phi(B)$ contains the factor $\phi_s(B) = U(B)$. The autoregressive polynomial of N_t, $\phi_N(B)$, must have no zeros in common with $U(B)$, otherwise N_t would contain a seasonal component. Thus,

$$\phi(B) = U(B)\phi_N(B) \tag{2.7}$$

which determines $\phi_N(B)$ from $\phi(B)$, and the relationship (2.4) becomes

$$\sigma_a^2\frac{\theta(B)\theta(F)}{\phi(B)\phi(F)} = \sigma_b^2\frac{\eta_s(B)\eta_s(F)}{U(B)U(F)} + \sigma_c^2\frac{\eta_N(B)\eta_N(F)}{\phi_N(B)\phi_N(F)} \tag{2.8}$$

It remains to determine the moving average polynomials $\eta_s(B)$ and $\eta_N(B)$, and the innovation variances σ_b^2 and σ_c^2. Any choice of moving average polynomials and variances satisfying (2.8), subject to the restriction that $\eta_s(B)$ is at most of degree 11, will be an *acceptable decomposition.*

To determine an acceptable decomposition, a partial fractions expansion of the left-hand side of (2.8) may be performed to yield

$$\sigma_a^2\frac{\theta(B)\theta(F)}{\phi(B)\phi(F)} = \frac{Q_s(B)}{U(B)U(F)} + \frac{Q_N(B)}{\phi_N(B)\phi_N(F)} \tag{2.9}$$

where $Q_s(B) = q_{0s} + \sum_{j=1}^{10} q_{js}(B^j + F^j)$ and $Q_N(B)$ may be obtained by subtraction. The expansion (2.9) is unique because the order of $Q_s(B)$ is less than that of $U(B)U(F)$. In the case where an acceptable decomposition exists, if we set $B = e^{-i\omega}$ in (2.8), both terms on the right-hand side are nonnegative for $0 \leq \omega \leq \pi$. Thus, to achieve an acceptable decomposition from the partial fractions expansion (2.9), we may need to modify the terms on the right-hand side so they are nonnegative for $0 \leq \omega \leq \pi$, while their sum remains the same. Because the degree of $\eta_s(B)$ is restricted to be at most 11, the only possible modifications are the addition of a constant, γ, to the first term and the sub-subtraction of γ from the second term. It follows from (2.9) by letting

$$\epsilon_1 = \min_{0\leq\omega\leq\pi} \frac{Q_s(e^{-i\omega})}{|U(e^{-i\omega})|^2} \text{ and } \epsilon_2 = \min_{0\leq\omega\leq\pi} \frac{Q_N(e^{-i\omega})}{|\phi_N(e^{-i\omega})|^2}$$

that an acceptable decomposition exists if and only if

$$\epsilon_1 + \epsilon_2 \geq 0.$$

If $\epsilon_1 + \epsilon_2 > 0$, then the acceptable decomposition is not unique because there is an interval for the constant γ such that both

$$\frac{Q_s(e^{-i\omega})}{|U(e^{-i\omega})|^2} + \gamma \text{ and } \frac{Q_N(e^{-i\omega})}{|\phi_N(e^{-i\omega})|^2} - \gamma$$

are nonnegative. Thus, when $\epsilon_1 + \epsilon_2 > 0$, the prior knowledge about S_t used to this point and the restrictions upon S_t imposed by the model for Z_t are not sufficient to determine a unique model for S_t. In this case, we must further restrict the model for S_t based upon additional a priori assumptions about the seasonal component.

2.3. Canonical Decomposition and Justifications

Following the ideas originally given in Tiao and Hillmer (1978), Box, Hillmer, and Tiao (1978), Pierce (1978), and Burman (1980), we define a canonical decomposition of Z_t into S_t and N_t as follows. Within the range of choices of $\eta_s(B)$, $\eta_N(B)$, σ_b^2, and σ_c^2 satisfying (2.8), the *canonical decomposition* is that one which minimizes σ_b^2, the innovation variance for the seasonal component. This defining property is intuitively pleasing since the randomness of S_t arises from the sequence of b_t's. Thus, minimizing σ_b^2 selects the model for the seasonal component which is as deterministic as possible while remaining consistent with the information in the data.

Some additional properties of the canonical decomposition can be cited. (See Hillmer and Tiao 1982.) (i) Among the set of all acceptable decompositions, the canonical decomposition minimizes Var$[U(B)S_t]$. This is appealing since making Var$[U(B)S_t]$ small, combined with the fact that $E[U(B)S_t] = 0$, will ensure that the sum of S_t over any 12 consecutive months remains close to zero. (ii) If $\bar{S}_t$ denotes the canonical seasonal component and $\tilde{S}_t$ denotes any other acceptable seasonal component, then $\tilde{S}_t = \bar{S}_t + e_t$ where e_t is a white noise series with variance $\sigma_e^2 > 0$. In other words, every acceptable seasonal component may be viewed as the sum of the canonical seasonal and white noise. To define the seasonal component to be $\tilde{S}_t$ seems unreasonable, since $\bar{S}_t$ is a highly predictable component which accounts for all the seasonality in Z_t, while e_t is nonseasonal and completely unpredictable. Therefore, in the absence of other information about S_t, it is reasonable to define the seasonal component to be $\overline{S}_t$.

We believe that the canonical decomposition is an appropriate choice. If, however, there was a priori knowledge about S_t leading to a different acceptable decomposition, that choice could be justified. It is important to note that there is not enough information in the data to uniquely determine the model for S_t, so that some additional defining assumptions about the seasonal component must be made in order to carry out the seasonal adjustment. It is a strength of the model-based approach that this fact is emphasized and that the assumptions being made are clearly specified.

2.4. Consideration of Some Special Problems

To be complete, we discuss some special problems that may arise. First, we have defined the canonical components only when $\phi(B)$ can be written as $U(B)\phi_N(B)$. While this covers many cases, it does not include all models that might be fit to seasonal data. One case that arises is where $\phi(B) = (1 - \phi_{12}B^{12})\phi^*(B)$ with $|\phi_{12}| < 1$ and $\phi^*(B)$ nonseasonal. In this case, we would not decompose Z_t because, unless ϕ_{12} is very near 1, the annual pattern of an estimate of S_t that might be produced will probably change very rapidly. This behavior does not correspond well to the general idea of what a seasonal component or its estimate should look like. Generally speaking, we would recommend seasonally adjusting a series only when $\phi(B)$ contains $U(B)$.

In practice, $U(B)$ enters into the model through seasonal differencing, i.e. $1 - B^{12} = (1 - B)U(B)$. After applying $1 - B^{12}$, we have found it more appropriate to account for remaining seasonality in the model through $\theta(B)$, e.g. $\theta(B) = (1 - \theta_{12}B^{12})\,\theta^*(B)$ where $\theta^*(B)$ is nonseasonal, rather than with additional seasonal autoregressive terms. If θ_{12} is small (say ≤ 0.4) this choice might be made for convenience since we could well approximate $(1 - .4B^{12})^{-1}$ with a seasonal autoregressive operator of low order. However, θ_{12} is typically much larger than 0.4. Thus, we have not found it necessary to deal with seasonal autoregressive operators apart from $U(B)$.

A final consideration involves series for which the seasonality is thought to be fixed, either from modeling the data or from a priori considerations. In this case, it is easy to estimate and remove S_t. (See Pierce 1978.) One can fit monthly means to Z_t, or more frequently to $(1 - B)Z_t$, constrain these to sum to zero, and subtract them out. We do not advocate subtracting monthly means from all series being adjusted, although this approach has been suggested by Pierce (1978) and Cleveland, Dempster, and Stith (1980).

2.5. Example

As an example to illustrate the ARIMA model-based seasonal adjustment approach, we consider the monthly time series of employed males aged 16 to 19 in nonagricultural industries from January 1965 to August 1979. This series was obtained from the Bureau of Labor Statistics and is given in the appendix. The data are plotted in figure 2.1a. Judging from this plot, it is evident that this series is seasonal, the level of the series is changing over time, and the variability over time is relatively stable. The sample autocorrelation functions of the series and selected differences are plotted in figures 2.2 through 2.5. Examination of the sample ACF's suggests regular and seasonal differences of the series to achieve stationarity. The sample ACF of the differenced series $W_t = (1 - B)(1 - B^{12})Z_t$ indicates that the model

$$(1 - B)(1 - B^{12})Z_t = (1 - \theta_1 B)(1 - \theta_{12}B^{12})a_t \qquad (2.10)$$

may be appropriate for this series. The parameters in model (2.10) were estimated (using the TSPACK program of Liu 1979) and the estimates are reported in table 2.1. The residual autocorrelations are plotted in figure 2.6 and the standardized residuals are plotted in figure 2.7. These plots reveal no model inadequacies. In addition, the Ljung-Box (1978) test statistic for overall model fit based upon 36 lags equals 39.7. This is less than 48.6, the level .05 chi-squared critical value with 34 degrees of freedom. Thus, (2.10) appears to be an adequate model for this data.

Table 2.1. PARAMETER ESTIMATES FOR MODEL (2.10)

Parameter	Estimate	Standard error
θ_1	0.27	0.073
θ_{12}	.82	.037

Assuming that the parameter estimates reported in table 2.1 are the true values, the theory of section 2.3 can be applied to estimate the canonical seasonal and nonseasonal components for this example. The estimated canonical nonseasonal and seasonal components are plotted in figures 2.1a and 2.1b. The nonseasonal component captures the underlying movements of the series and the seasonality in this series is relatively stable.

3. TRADING-DAY AND HOLIDAY VARIATION

It is not unusual for monthly economic time series to be affected by the composition of the calendar. The two primary calendar influences are trading-day effects and holiday effects. In this section, we summarize the results in Bell and Hillmer (1981) on modeling series containing these effects.

3.1. Trading-Day Variation

Suppose the level of retail sales in a type of business (e.g. grocery stores) is greater on Friday and Saturday than on other days of the week. Over the years the same month, say January, will contain a different number of Fridays and Saturdays so that the level of retail sales for January in a given year will be affected by the number of Fridays and Saturdays in that

Figures 2.1-2.7 EXAMPLE OF ARIMA MODEL-BASED SEASONAL ADJUSTMENT: EMPLOYED MALES, 16-19, IN NONAGRICULTURAL INDUSTRIES

Figure 2.1a ORIGINAL AND ADJUSTED SERIES

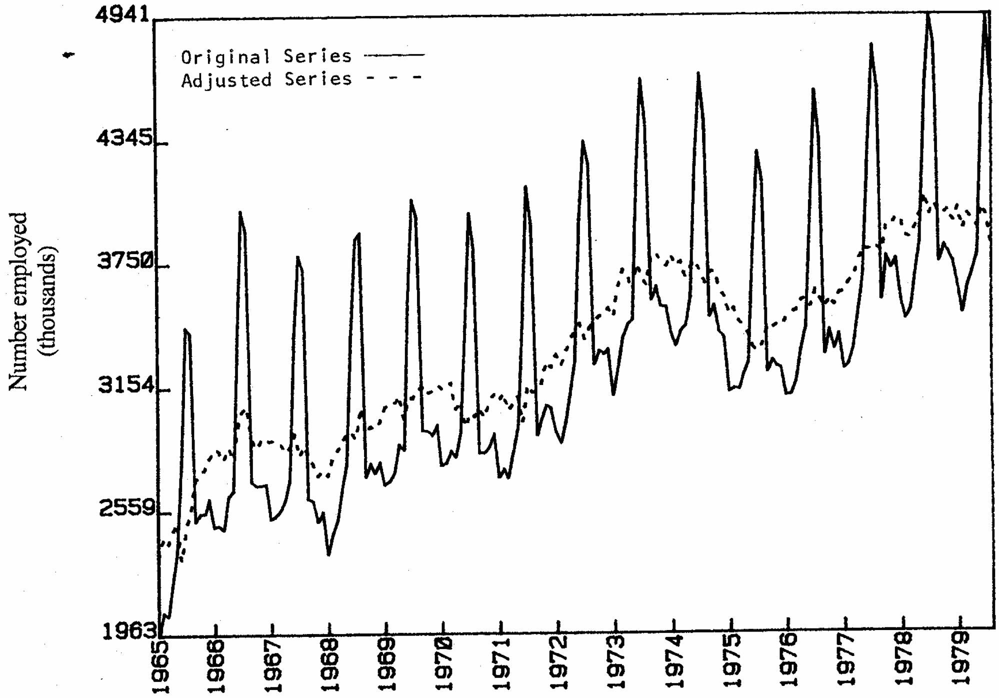


Figure 2.1b SEASONAL FACTORS

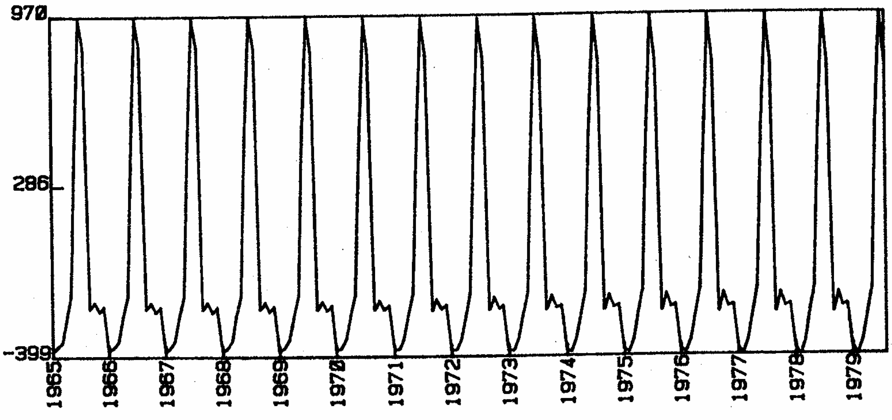


Figure 2.2 SACF OF Z_t

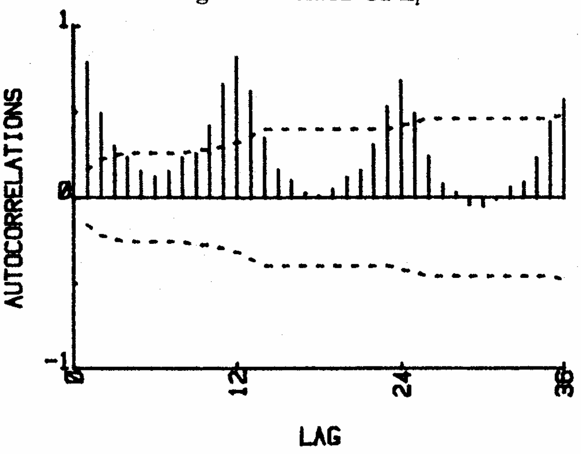


Figure 2.3 SACF of $(1 - B)Z_t$

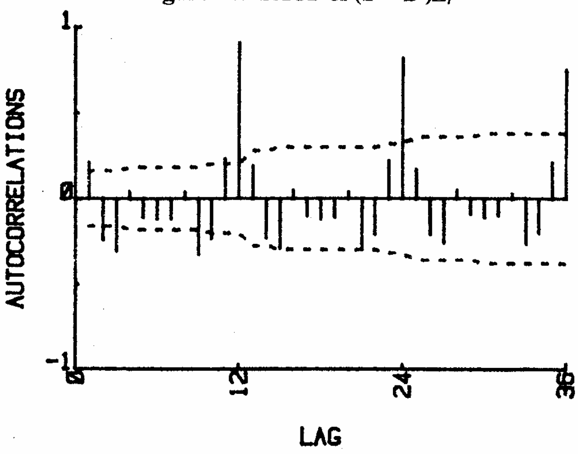


Figure 2.4 SACF of $(1 - B^{12})Z_t$

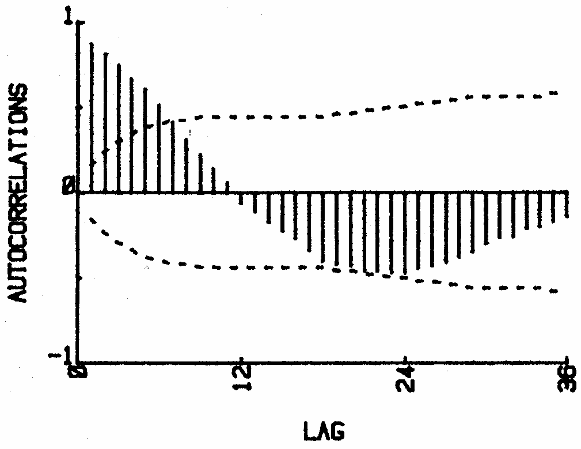


Figure 2.5 SACF of $(1 - B)(1 - B^{12})Z_t$

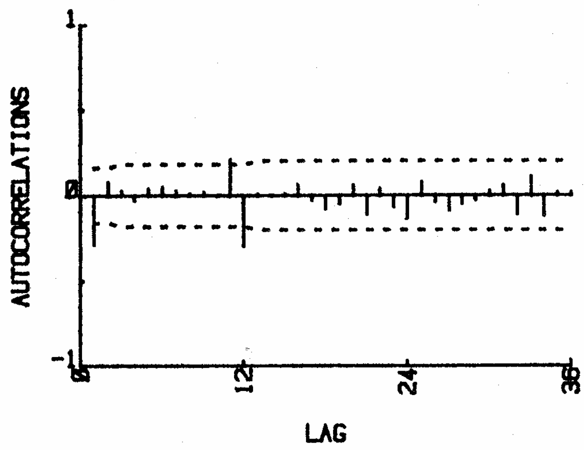


Figure 2.6 SACF OF RESIDUALS FROM MODEL (2.10)

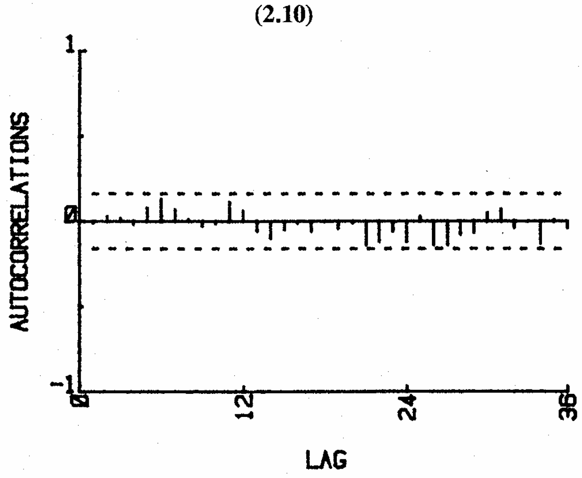
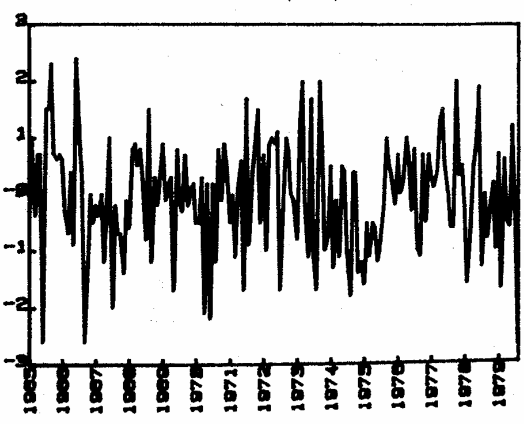


Figure 2.7 STANDARDIZED RESIDUALS FROM MODEL (2.10)



particular January. The variation in a monthly time series that is related to the day-of-the-week composition of the calendar is called trading-day variation. We might expect that economic time series on sales, production, shipments, monetary activity, and service activity may all be subject to trading-day variation. Thus, it is important to be able to deal with this in modeling and seasonally adjusting these time series.

In modeling series with trading-day variation, we suppose that the trading-day effect can be approximated by a deterministic model. Let TD_t denote the trading-day factor for month t; then TD_t will be a function of the number of distinct types of days in month t. In particular, we suppose that

$$TD_t = \sum_{i=1}^{7} \gamma_i X_{it} \qquad (3.1)$$

where X_{it}, $i = 1, \ldots, 7$, are respectively the number of Mondays, Tuesdays, Wednesdays, Thursdays, Fridays, Saturdays, and Sundays in month t, and γ_i, $i = 1, \ldots, 7$, are parameters. The model (3.1) is appropriate for flow series, such as sales, where the monthly values can be thought of as the accumulation of daily values. For stock series, other approaches should be considered. (See, for example, Cleveland and Grupe 1982.)

The model (3.1) can be written as

$$TD_t = \sum_{i=1}^{6} (\gamma_i - \bar{\gamma})(X_{it} - X_{7t}) + \bar{\gamma} \sum_{i=1}^{7} X_{it} \qquad (3.2)$$
$$= \sum_{i=1}^{7} \beta_i T_{it}$$

where $\bar{\gamma} = 1/7 \sum_{i=1}^{7} \gamma_i$, $\beta_i = \gamma_i - \bar{\gamma}$ and $T_{it} = X_{it} - X_{7t}$ for $i = 1, \ldots, 6$, $\beta_7 = \bar{\gamma}$, and $T_{7t} = \sum_{i=1}^{7} X_{it}$ denotes the length of month t. The parameterization (3.2) is more convenient than (3.1) because estimates of the γ_i's tend to be highly correlated while estimates of the β_i's are less correlated. Also, when making inferences, the differential effects $\beta_1, \ldots, \beta_6$ are of more interest than the γ_i's.

In addition to the trading-day variation characterized by (3.2), we must also deal with autocorrelation, trends, and seasonality. To do this, we assume that apart from trading-day effects the series follows an ARIMA model. Thus, letting Z_t^* denote the value of an observed time series at month t including trading-day effects, an additive model for Z_t^* is

$$Z_t^* = TD_t + Z_t \qquad (3.3)$$

where TD_t is defined by (3.2) and Z_t follows (2.3).

Modeling strategy. In building models of the form (3.3), the following approach has been found effective.

(i) Direct identification of the ARIMA model for Z_t is often difficult because of the confounding of the autocorrelation pattern of Z_t with the influences of TD_t. However, one can typically determine the appropriate degree of differencing (d and D of $(1 - B)^d (1 - B^{12})^D$ in the model for Z_t) by examining the sample autocorrelation function of Z_t^* and its differences in the usual way.

(ii) Given d and D, the trading-day effects can be approximately removed by regressing $(1 - B)^d (1 - B^{12})^D Z_t^*$ on $(1 - B)^d (1 - B^{12})^D T_{it}$, $i = 1, \ldots, 7$. The model for Z_t can then be identified by examining the sample autocorrelation function (SACF) and sample partial autocorrelation function (SPACF) of the residuals from this regression.

(iii) Once a model for Z_t has been tentatively identified, maximum likelihood estimates of the trading-day and time series parameters can be computed, and the results of Pierce (1971) can be used to make inferences.

(iv) Standard diagnostic checks described in Box and Jenkins (1970) can be used to assess model inadequacy and suggest directions of improvement.

Removal of estimated trading-day variation in seasonal adjustment. Given a model for Z_t^*, the estimated trading-day effects are $\hat{TD}_t = \sum_{i=1}^{7} \hat{\beta}_i T_{it}$ where $\hat{\beta}_i$ $i = 1, \ldots, 7$ are the estimated trading-day parameters. For seasonal adjustment, it is desirable that the long-run average of the trading-day adjustment factors be zero. Now

$$\frac{1}{n} \sum_{t=1}^{n} \sum_{i=1}^{6} \beta_i T_{it} = \sum_{i=1}^{7} (\gamma_i - \bar{\gamma})[\frac{1}{n} \sum_{t=1}^{n} (X_{it} - X_{7t})] \doteq 0$$

for large n since the $1/n \sum_{t=1}^{n} (X_{it} - X_{7t})$ are approximately constant while $\sum_{i=1}^{7} (\gamma_i - \bar{\gamma}) = 0$. Furthermore, we can write

$$T_{7t} = (T_{7t} - LF_t - 30.4375) + LF_t + 30.4375$$

where

$$LF_t = \begin{cases} .75 & \text{for a February in a leap year} \\ -.25 & \text{for a February in a nonleap year} \\ 0 & \text{otherwise} \end{cases}$$

and

$$30.4375 = \frac{365.25}{12}$$

It can be easily seen that $(T_{7t} - LF_t - 30.4375)$ sums to zero over any 12 consecutive months and LF_t sums to zero over any 48 consecutive months. Thus, from

$$TD_t = \sum_{i=1}^{6} \beta_i T_{it} + \beta_7 LF_t +$$
$$\beta_7(T_{7t} - LF_t - 30.4375) + \beta_7(30.4375) \qquad (3.4)$$

we see the sum of the first two terms on the right-hand side of (3.4) is the trading-day adjustment factor, while the third term is part of the seasonal component and the fourth is part of the nonseasonal component.

To seasonally adjust a series with trading-day effects, we first form $\dot{Z}_t = Z_t^* - \overline{TD}_t$ where $\overline{TD}_t = \sum_{i=1}^{6} \hat{\beta}_i T_{it} + \hat{\beta}_7 LF_t$. Then we apply the results of section 2 by computing $\hat{N}_t = W_N(B)\dot{Z}_t$ and $\hat{S}_t = W_s(B)\dot{Z}_t$. When this is done, the deterministic seasonal effect $\hat{\beta}_7(T_{7t} - LF_t - 30.4375)$ is automatically assigned to $\hat{S}_t$ since $W_N(B) + W_s(B) = 1$ and $W_N(B)[T_{7t} - LF_t - 30.4375] = 0$ because $U(B)[T_{7t} - LF_t - 30.4375] = 0$. (See (2.4) and (2.5) and recall $\phi_s(B) = U(B)$.) Also, assuming $\phi(B)$ contains $1 - B$ (in most applications it contains the factor $1 - B^{12} = (1 - B)U(B)$), $\hat{\beta}_7(30.4375)$ is automatically assigned to $\hat{N}_t$ since then $\phi_N(B)$ contains $1 - B$ and $W_s(B)(30.4375) = 0$ because $\phi_N(B)(30.4375) = 0$. An estimate of the combined trading-day and seasonal component is $\overline{TD}_t + \hat{S}_t$.

Trading-day example. To illustrate the ideas in this section, we analyze the monthly series of U.S. wholesale sales of hardware, plumbing, heating equipment, and supplies from January 1967 through November 1979. The data were obtained from the Census Bureau and are given in the appendix. The data are plotted in figure 3.11a. From the plot of the data, it is apparent that the series is seasonal, the level increases over time, and the variability increases with the level. To stabilize the variability we took logarithms of the data; these are plotted in figure 3.1.

The SACF's of the logged series and selected differences are plotted in figures 3.2 to 3.5. We conclude that a first difference and possibly a twelfth difference are necessary to induce stationarity. The SACF of the first and twelfth differenced series exhibits a complex pattern. Analysts at the Census Bureau indicated that this series is influenced by trading-day variation; consequently, the pattern in figure 3.5 may be due to confounding by the trading-day effects. To approximately remove the trading-day variaton, the model

$$(1 - B)(1 - B^{12})Z_t^* = \sum_{i=1}^{7} \beta_i (1 - B)(1 - B^{12})T_{it} + N_t \tag{3.5}$$

was fit by least squares to the logged data, Z_t^*. The SACF of the residuals from (3.5) is plotted in figure 3.6. The most pronounced feature is the negative spike at lag 12 suggesting the model

$$Z_t^* = \sum_{i=1}^{7} \beta_i T_{it} + \frac{(1 - \theta_{12}B^{12})}{(1 - B)(1 - B^{12})} a_t \tag{3.6}$$

Note that before allowing for trading-day effects, the autocorrelation function (figure 3.3) does not necessarily indicate that twelfth differencing is needed since the autocorrelations at multiples of lag 12 die out relatively quickly. Assuming we only need to first difference, the trading-day effects may be approximately removed by consideration of the residuals from the least squares fit of the model

$$(1 - B)Z_t^* = \sum_{i=1}^{7} \beta_i (1 - B)T_{it} + N_t \tag{3.7}$$

The persistence of the autocorrelations of these residuals (plotted in figure 3.7) at multiples of lag 12 now indicates that a seasonal difference is required to obtain a stationary noise model.

After estimation of the model (3.6), examination of the SACF of the residuals, plotted in figure 3.8, reveals a significant negative spike at lag 1. Thus, we are led to consider the model

$$Z_t^* = \sum_{i=1}^{7} \beta_i T_{it} + \frac{(1 - \theta_1 B)(1 - \theta_{12}B^{12})}{(1 - B)(1 - B^{12})} a_t \tag{3.8}$$

We see from the parameter estimates and t-ratios for model (3.8), reported in table 3.1, that $\hat{\theta}_1$, $\hat{\theta}_{12}$, $\hat{\beta}_2$, $\hat{\beta}_4$, $\hat{\beta}_6$, and $\hat{\beta}_7$ are all statistically significant. The estimated trading-day parameters $\hat{\beta}_1, \ldots, \hat{\beta}_6$ indicate that sales are higher on Tuesdays and Thursdays while lower on weekends. (The estimated effect of Sunday, $\gamma_7 - \bar{\gamma}$, is $-.015$.) However, note from table 3.1 that the trading-day parameter estimates are not independent, so that individual inferences about these parameters must be made with care. Since at least one of the β_i's is significantly different from zero, all of the trading-day parameters will be retained for the purpose of seasonal adjustment.

Table 3.1. PARAMETER ESTIMATES AND CORRELATION MATRIX FOR (3.8)

Parameter	Estimate	t-ratio
θ_1	0.22	2.6
θ_{12}	.75	12.1
β_1	.001	.3
β_2	.013	3.5
β_3	.004	1.1
β_4	.011	3.0
β_5	.001	.2
β_6	−.015	−4.0
β_7	.026	2.0

Correlation Matrix

θ_1	1.00								
θ_{12}	.07	1.00							
β_1	.04	.03	1.00						
β_2	.02	.04	−.57	1.00					
β_3	−.03	−.04	−.05	−.53	1.00				
β_4	.03	.01	.11	−.02	−.55	1.00			
β_5	−.08	−.02	.02	.09	−.03	−.56	1.00		
β_6	.04	−.02	.02	.02	.09	.01	−.56	1.00	
β_7	.05	.02	.14	−.13	.13	−.13	.05	.09	1.00

As diagnostic checks upon the adequacy of model (3.8), the SACF of the residuals is given in figure 3.9 and the standardized residuals are plotted figure 3.10. The only possible concern is the series of negative residuals around the year 1975;

for our purposes, these were ignored. The Ljung-Box test statistic for overall model fit based upon 36 lags is 39.2, which is less than 48.6, the .05 chi-squared value with 34 degrees of freedom.

Assuming the model for the wholesale sales of hardware is (3.8) and the parameter estimates in table 3.1 are the true values, it is possible to compute estimates of the trading-day, canonical seasonal, canonical nonseasonal, and combined trading-day seasonal components. These are plotted in figures 3.11a to 3.11d, where the estimated nonseasonal component has been transformed to the original metric of sales by exponentiation, and the estimates of the other components have been exponentiated and multiplied by 100.

From figures 3.11c and 3.11d, we notice that the estimated canonical seasonal component is fairly stable over time, and the relative impact of the adjustment for the seasonal is greater than that for the trading day. The trading-day factors vary about 100 percent in an irregular fashion, which results in the combined trading-day seasonal component (fig. 3.11b) showing much more erratic behavior than the canonical seasonal. Finally, the estimated nonseasonal component seems to follow the underlying movements of the original data.

Figures 3.1-3.11 EXAMPLE OF SEASONAL AND TRADING-DAY ADJUSTMENT: WHOLESALE SALES OF HARDWARE

Figure 3.1 LOG WHOLESALE SALES (Z_t^*)

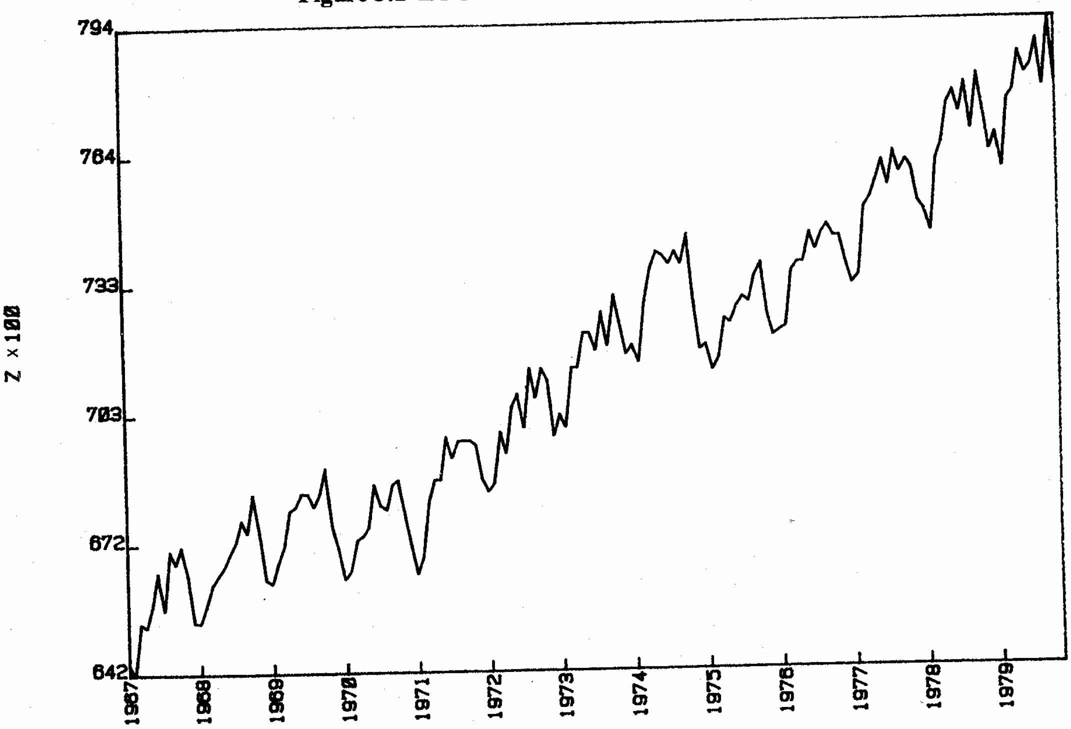


Figure 3.2 SACF OF Z_t^*

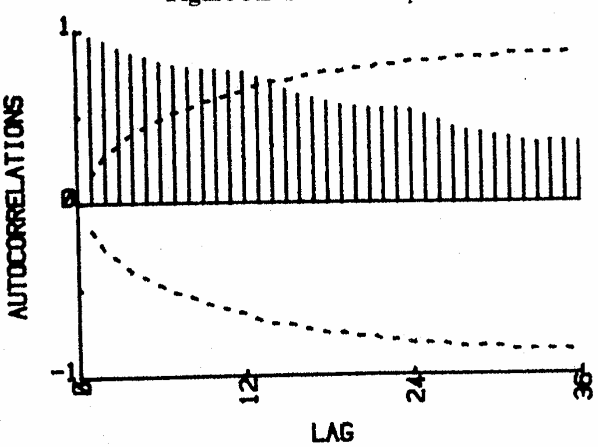


Figure 3.3 SACF OF $(1 - B)Z_t^*$

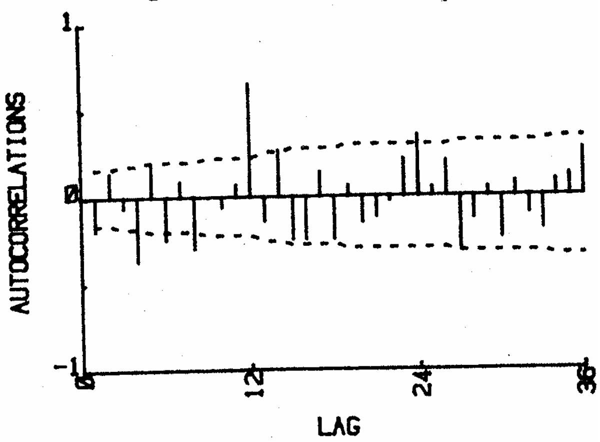


Figure 3.4 SACF OF $(1 - B^{12})Z_t^*$

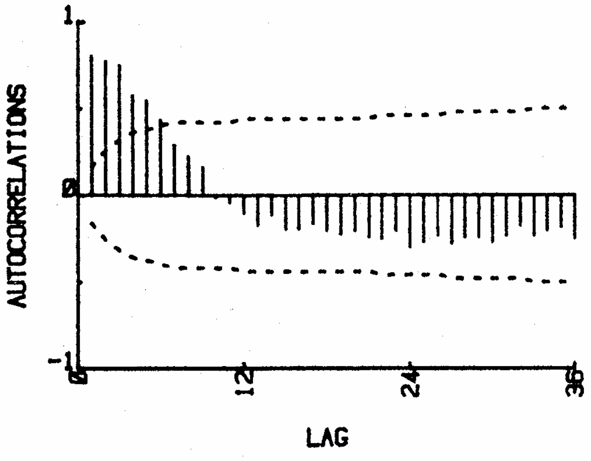


Figure 3.5 SACF OF $(1 - B)(1 - B^{12})Z_t^*$

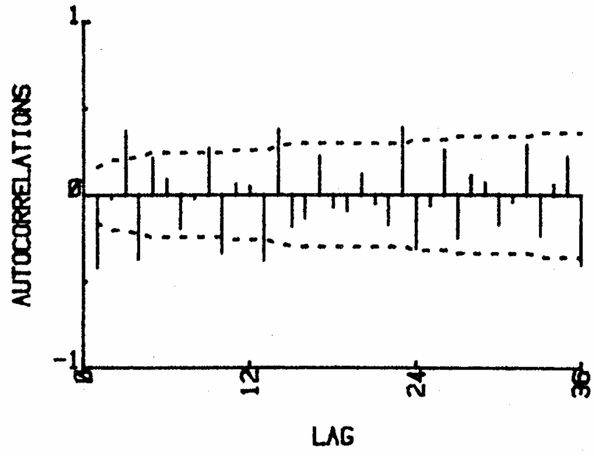


Figure 3.6 SACF OF RESIDUALS FROM MODEL (3.5)

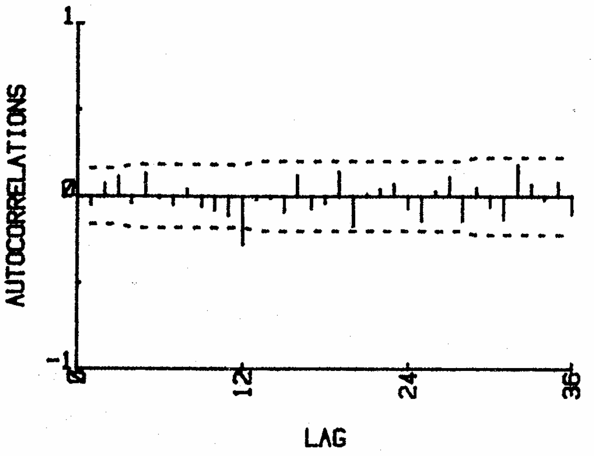


Figure 3.7 SACF OF RESIDUALS FROM MODEL (3.7)

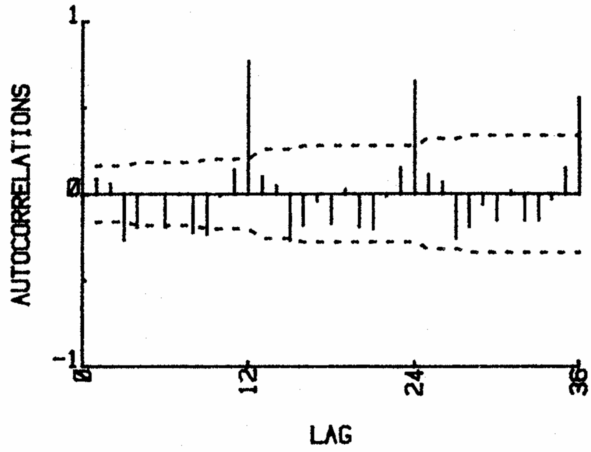


Figure 3.8 SACF OF RESIDUALS FROM MODEL (3.6)

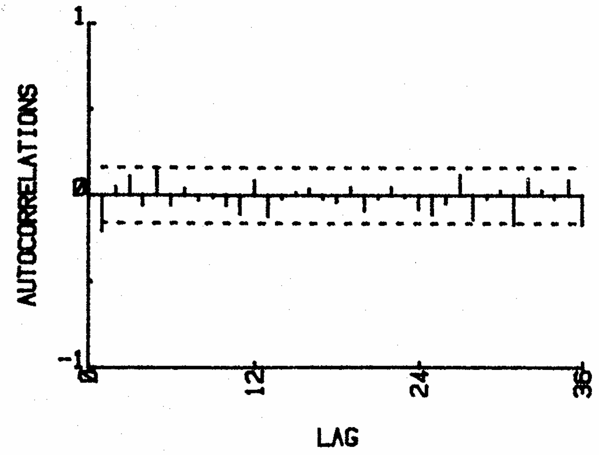


Figure 3.9 SACF OF RESIDUALS FROM MODEL (3.8)

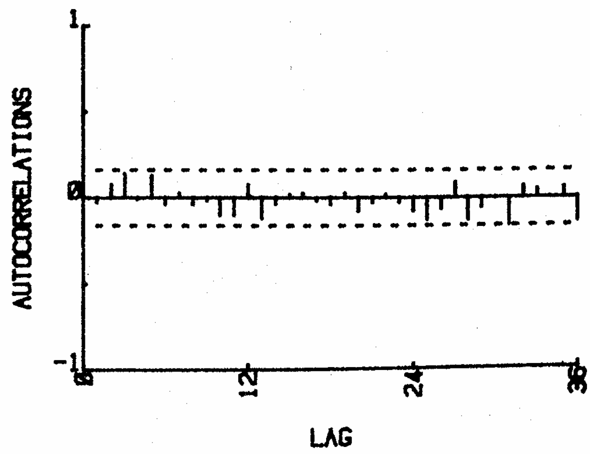
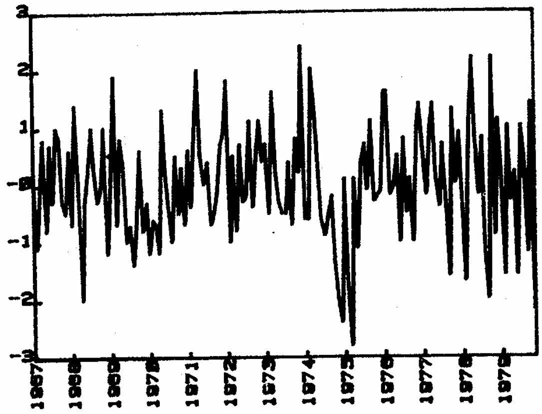


Figure 3.10 STANDARDIZED RESIDUALS FROM MODEL (3.8)



3.2. Easter Holiday Variation

Some economic time series are affected by holidays that recur each year at different times. The principle example of this for U.S. series is the increase in the level of some retail sales series in the days preceding Easter. Because Easter occurs at various dates during March and April, the monthly values for these months can be affected by the date of Easter each year. When the placement of holidays impacts the level of a series, it is important to develop models to account for these effects. Here we shall restrict attention to Easter holiday effects, although a similar approach could be used to model the effects of other holidays.

Series influenced by the placement of Easter may also exhibit trading-day variation. Let Z_t^* denote an appropriately transformed time series containing Easter and trading-day effects, and let E_t denote the Easter effect for month t. Then an additive model for Z_t^* is

$$Z_t^* = E_t + TD_t + Z_t \tag{3.9}$$

where TD_t is given by (3.2) and Z_t by (2.3).

Specifying a functional form for E_t is not as easy as for TD_t because the placement of Easter affects daily sales for a period shorter than a month prior to Easter, while we typically have monthly data. As a first approximation, suppose there is a constant increase $\tilde{\alpha}$ in sales each day for τ days before Easter. Let $H(\tau,\ t)$ denote the proportion of the time period τ days before Easter that falls in month t. Then the Easter effect is

$$E_t = \tilde{\alpha}[\tau H(\tau,\ t)] = \alpha H(\tau,\ t) \tag{3.10}$$

where $\alpha = \tilde{\alpha}\tau$.

The effect of the placement of Easter need not follow the pattern assumed in (3.10). However, a limited number of Easter dates will occur during the time frame of any series, and this limits the type of Easter effect that can be estimated. Bell and Hillmer (1981) show a way to test if the simple effect in

Figure 3.11a WHOLESALE SALES OF HARDWARE, ORIGINAL AND ADJUSTED SERIES

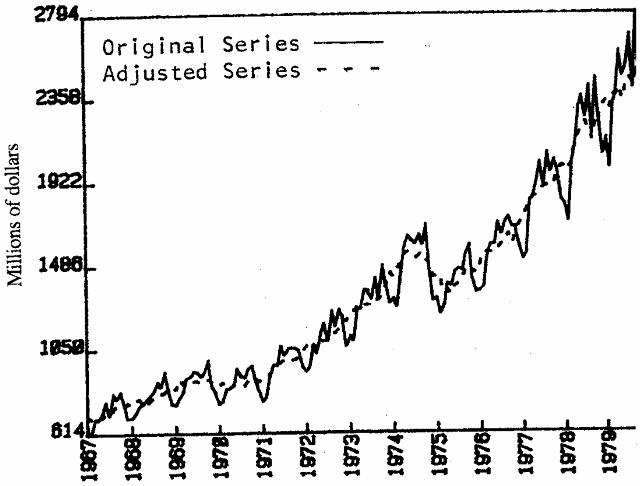


Figure 3.11b COMBINED SEASONAL AND TRADING-DAY FACTORS

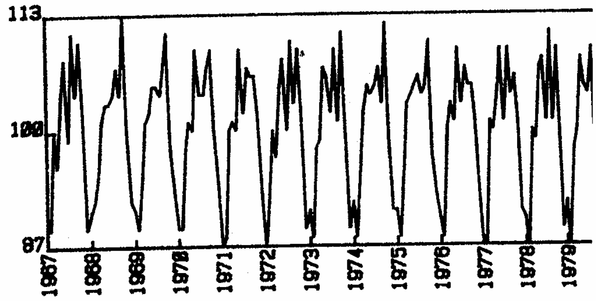


Figure 3.11c SEASONAL FACTORS

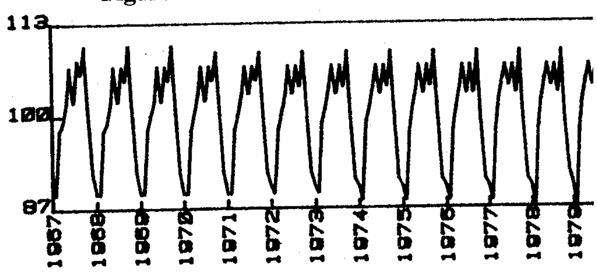
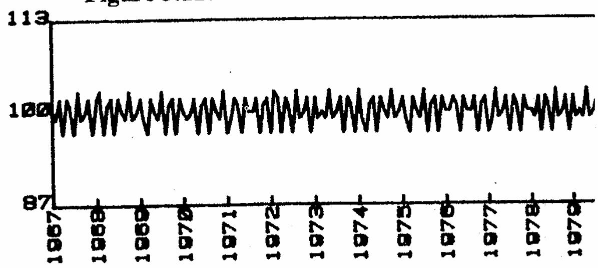


Figure 3.11d TRADING-DAY FACTORS



(3.10) is reasonable for a given series and discuss more general effects.

Modeling strategy. To model Easter variation, we proceed in a manner similar to that discussed for trading-day variation. Thus, we (i) identify the degree of differencing, $(1-B)^d(1-B^{12})^D$, from the SACF of the original series and its differences, (ii) remove the trading-day and Easter effects in a preliminary fashion to allow a model for $Z_t = Z_t^* - TD_t - E_t$ to be identified, and (iii) efficiently estimate and check the entire model. To remove E_t and TD_t in (ii) and identify a model for Z_t, we make a specific choice of τ in (3.10) (e.g. $\tau = 14$), regress $(1-B)^d(1-B^{12})^D Z_t^*$ on $(1-B)^d(1-B^{12})^D H(14, t)$ and $(1-B)^d(1-B^{12})^D T_{it}$, $i = 1, \ldots, 7$, and examine the SACF and SPACF of the residuals from this regression.

Parameter estimation. The model is (3.9) with E_t given by (3.10) and TD_t by (3.2), so

$$Z_t^* = \alpha H(\tau, t) + \sum_{i=1}^{7} \beta_i T_{it} + Z_t \tag{3.11}$$

Assuming a suitable ARIMA model has been identified for Z_t, we wish to estimate the parameters of the model for Z_t^* including τ. Gaussian maximum likelihood estimates (MLE's) of the parameters can be obtained as follows.

Let $L(\alpha, \tau, \underset{\sim}{\beta}, \underset{\sim}{\phi}, \underset{\sim}{\theta}, \sigma_a^2)$ denote the log-likelihood function, where $\underset{\sim}{\beta} = (\beta_1, \ldots, \beta_7)'$, $\underset{\sim}{\phi} = (\phi_1, \ldots, \phi_p)'$, and $\underset{\sim}{\theta} = (\theta_1, \ldots, \theta_q)'$. Maximizing this over α, $\underset{\sim}{\beta}$, $\underset{\sim}{\phi}$, $\underset{\sim}{\theta}$, and σ_a^2 for fixed τ gives (asymptotically)

$$L_{\max}(\tau) = \max_{\alpha, \underset{\sim}{\beta}, \underset{\sim}{\phi}, \underset{\sim}{\theta}, \sigma_a^2} L(\alpha, \tau, \underset{\sim}{\beta}, \underset{\sim}{\phi}, \underset{\sim}{\theta}, \sigma_a^2) = -\frac{n}{2} \ln \hat{\sigma}_a^2(\tau)$$

where $\hat{\sigma}_a^2(\tau)$ is the MLE of σ_a^2. Fitting (3.11) for a suitable set of τ's and picking the τ that minimizes $\hat{\sigma}_a^2(\tau)$ gives $\hat{\tau}$, the MLE of τ. The estimates $\hat{\alpha}$, $\hat{\underset{\sim}{\beta}}$, $\hat{\underset{\sim}{\phi}}$, $\hat{\underset{\sim}{\theta}}$, $\hat{\sigma}_a^2(\tau)$ from the fit with $\hat{\tau}$ are the MLE's of α, $\underset{\sim}{\beta}$, $\underset{\sim}{\phi}$, $\underset{\sim}{\theta}$, and σ_a^2.

For fixed τ (3.11) is linear in α and $\underset{\sim}{\beta}$, so the results of Pierce (1971) may be used to make inferences conditional upon τ. Unconditional inferences are difficult to make because $H(\tau, t)$ is a nonlinear and nondifferentiable function of τ. In practice, due to the limited number of observed Easter dates, it is unlikely that τ can be estimated with great accuracy, so that a range of values for τ will yield broadly similar estimates of the other parameters. In this case, inferences made conditional upon $\tau = \hat{\tau}$ should not be misleading. In any event, this can be checked by examining the estimates of the other parameters and their standard errors for various values of τ.

Seasonally adjusting series with Easter effects. The Census Bureau's adjustment for Easter is done in the context of a multiplicative model and uses the reciprocal of the March adjustment factor for April. We use an additive decomposition on a suitably transformed series so a similar requirement imposed on our Easter factors would be that they sum to zero. So far, we have accounted for the effect of Easter by including $E_t = \alpha H(\tau, t)$ in our model without worrying about this restriction.

Notice that $H(\tau, t)$ sums to 1 over any calendar year. We assume $\tau \leqslant 21$; then $H(\tau, t)$ is nonzero only for March and April. We let MA_t equal 1 in March and April and 0 otherwise and write

$$E_t = \alpha[H(\tau, t) - \frac{1}{2}MA_t] + \alpha[\frac{1}{2}MA_t - \frac{1}{12}] + \frac{\alpha}{12} \tag{3.12}$$

Now in (3.12), $[H(\tau, t) - \frac{1}{2}MA_t]$ is zero outside of March and April, and it sums to zero over March and April, thus satisfying the restrictions we desire for an Easter effect. Since $U(B)MA_t = 2$, $[\frac{1}{2}MA_t - \frac{1}{12}]$ in (3.12) sums to zero over any 12 consecutive months, a condition we desire of a seasonal effect. Thus, in (3.12) we let $\alpha[H(\tau, t) - \frac{1}{2}MA_t]$ be the Easter effect, $\alpha[\frac{1}{2}MA_t - \frac{1}{12}]$ be a seasonal effect, and $\frac{\alpha}{12}$ be part of the level of the series.[1]

To seasonally adjust a series with Easter and trading-day effects, we first remove the estimated Easter and trading-day effects and then apply the results of section 2. Thus, we form

$$\dot{Z}_t = Z_t^* - \hat{\alpha}[H(\hat{\tau}, t) - \frac{1}{2}MA_t] - \sum_{i=1}^{6} \hat{\beta}_i T_{it} - \hat{\beta}_7 LF_t$$

and compute $\hat{N}_t = W_N(B)\dot{Z}_t$ and $\hat{S}_t = W_s(B)\dot{Z}_t$. When this is done the deterministic seasonal effect, $\hat{\alpha}[\frac{1}{2}MA_t - \frac{1}{12}]$, is automatically assigned to $\hat{S}_t$ since $W_N(B) + W_s(B) = 1$ and $W_N(B)[1/2MA_t - 1/12] = 0$. (See (2.5).) Similarly, $\hat{\alpha}/12$ is automatically assigned to $\hat{N}_t$.

Holiday example. We consider the time series of monthly retail sales of U.S. men's and boys' clothing stores from January 1967 through September 1979. The data is given in the appendix. The plot of the original data in figure 3.21a shows that it increases in level over time and that the seasonal amplitude varies with the level. Taking logarithms (fig. 3.12) seems to stabilize the seasonal amplitude. SACF's of the log series Z_t^*, and of $(1-B)Z_t^*$, $(1-B^{12})Z_t^*$, and $(1-B)(1-B^{12})Z_t^*$, are given in figures 3.13-3.16. We need to take $(1-B)(1-B^{12})Z_t^*$ to get the autocorrelations to die out.

The behavior of the SACF of $(1-B)(1-B^{12})Z_t^*$ at and near lags 36 and 48 is indicative of the Easter effect in this

[1]This effectively assumes that the long-run average Easter effect is $\alpha/2$ in both March and April, whereas it really depends on τ. A more refined approach would be to replace MA_t with the long-run average of $H(\tau, t)$ for each month.

series as pointed out by Bell and Hillmer (1981). Figures 3.17 and 3.18 present the SACF and SPACF of the residuals from a regression of $(1 - B)(1 - B^{12})Z_t^*$ on $(1 - B)(1 - B^{12})H(14, t)$ and $(1 - B)(1 - B^{12})T_{it}$, $i = 1, \ldots, 7$. Notice there are no longer spikes at lags 36 and 48 in the SACF. The SACF and SPACF suggest the tentative model

$$Z_t^* = \alpha H(\tau, t) + \sum_{i=1}^{7} \beta_i T_{it} + Z_t \qquad (3.13)$$

where

$$(1 - B)(1 - B^{12})Z_t = (1 - \theta_1 B - \theta_2 B^2)(1 - \theta_{12} B^{12})a_t$$

This model was fitted to the data with $\tau = 14$ and diagnostic checks revealed no model inadequacies.

We now estimate τ jointly with the other parameters in (3.13). This was done by estimating (3.13) for integer values of τ from 1 to 25. Some of the results are given in table 3.2. We see $100\hat{\sigma}_a^2(\tau)$ is minimized around $\hat{\tau} = 9$, although a wide range of τ values works about as well. The residual ACF for the model with $\tau = 9$ is shown in figure 3.19. It reveals no model inadequacies, and the Ljung-Box test statistic using 36 lags is 27.4, which is less than the χ^2_{33} 5-percent critical value

Table 3.2a. ESTIMATION OF τ

τ	1	2	3	4	5
$100\hat{\sigma}_a^2(\tau)$	.126	.122	.122	.123	.123
τ	6	7	8	9	10
$100\hat{\sigma}_a^2(\tau)$	.122	.121	.1203	.1201	.1205
τ	11	12	13	14	15
$100\hat{\sigma}_a^2(\tau)$	.122	.123	.125	.127	.127
τ	16	17	18	19	20
$100\hat{\sigma}_a^2(\tau)$	.128	.129	.129	.130	.131
τ	21	22	23	24	25
$100\hat{\sigma}_a^2(\tau)$	.131	.131	.131	.131	.131

Table 3.2b. ESTIMATION FOR $\tau = \hat{\tau} = 9$

Parameter	Estimate	t-ratio
θ_1	.26	3.2
θ_2	.37	4.5
θ_{12}	.78	15.5
β_1	−.010	−1.8
β_2	−.002	−.4
β_3	.005	1.1
β_4	−.002	−.3
β_5	.011	2.2
β_6	.013	2.5
β_7	.014	.8
α	.070	7.7

of 47.4. The standardized residuals, plotted in figure 3.20, do not indicate any problems with the model.

We can now estimate the canonical seasonal and nonseasonal components in the manner discussed earlier. The components were first estimated in the log-metric and then exponentiated. Figure 3.21a shows the original data and estimated nonseasonal component and figure 3.21b the estimate of the combined holiday trading-day seasonal component. The pattern in figure 3.21b varies from year to year, especially for the months of March, April, and December. Figures 3.21c to 3.21e give the estimated seasonal, trading-day, and holiday components, respectively. We notice from figure 3.21c that December has far and away the largest seasonal influence (due to Christmas) and that the pattern is fairly stable from year to year. The less regular pattern in figure 3.21b is due to the erratic patterns of the trading-day and holiday components.

4. DETECTION AND REMOVAL OF THE EFFECTS OF OUTLIERS

Economic and business time series observations are often subject to the influence of nonrepetitive exogenous interventions, e.g., strikes, outbreaks of wars, sudden changes in the market structure of a commodity, and unexpected heat or cold waves. When the timings of such interventions are known, their effects can often be accounted for in a model using intervention analysis techniques proposed by Box and Tiao (1975). As an illustration, again let Z_t^* be the observable time series, and suppose that an intervention occurs at time t_o. The effect can often be modeled as

$$Z_t^* = \frac{\omega(B)}{\delta(B)} \xi_t^{(t_0)} + Z_t \qquad (4.1)$$

where $$\xi_t^{(t_0)} = \begin{cases} 1 & \text{for } t = t_o \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

signifies the time of occurrence of the intervention, $\omega(B) = \omega_0 - \omega_1 B - \ldots - \omega_s B^s$, $\delta(B) = 1 - \delta_1 B - \ldots - \delta_r B^r$, and the ratio $\omega(B)/\delta(B)$ describes the dynamic behavior of the intervention.

In practice, the timings of exogenous interventions are often unknown to the statistical analyst. Since the effects of the interventions can lead to bias in parameter estimates, and hence in forecasts and seasonal adjustments, it is important to develop procedures which can help detect and remove such effects. This has come to be known as the problem of "outliers" or "spurious observations." In the Census X−11 procedure, outliers are handled in the filtering process. However, we believe that it is best to treat this problem as another element in the modeling process. If the time t_0 is known, this problem is not different in principle from that of modeling trading-day or holiday variation. For the situation when t_0 is unknown, we summarize the results on outliers in time series of Chang and Tiao (1982), following earlier work by Fox (1972).

Figures 3.12-3.21 EXAMPLE OF SEASONAL, TRADING-DAY, AND HOLIDAY ADJUSTMENT: RETAIL SALES OF MEN'S AND BOYS' CLOTHING

Figure 3.12 LOG RETAIL SALES (Z_t^*)

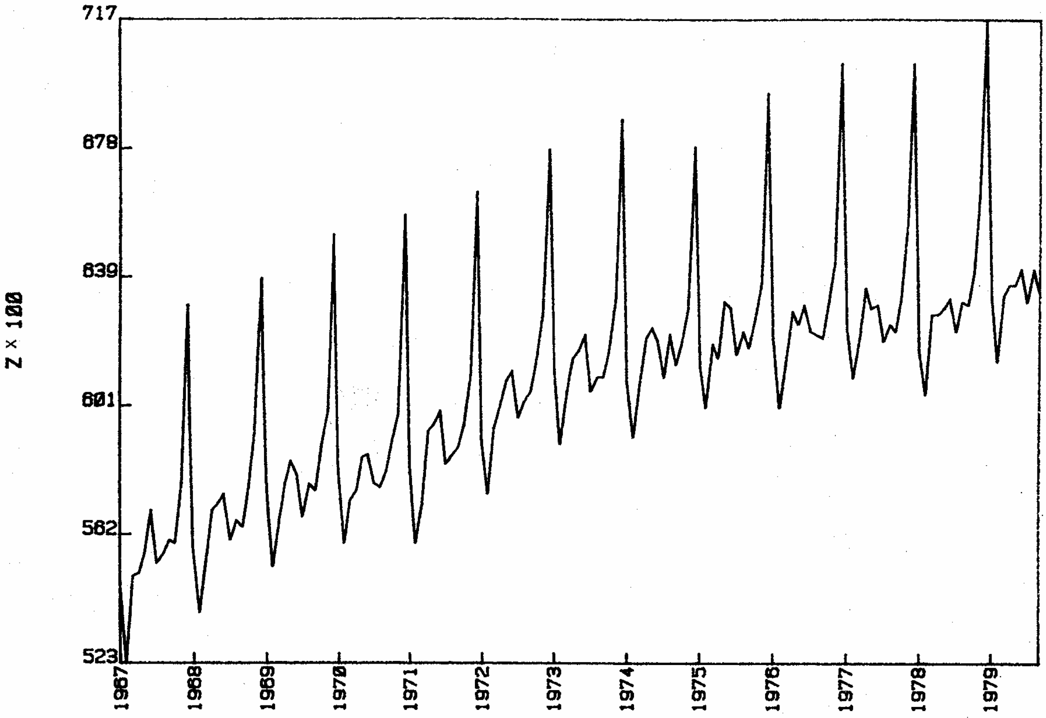


Figure 3.13 SACF OF Z_t^*

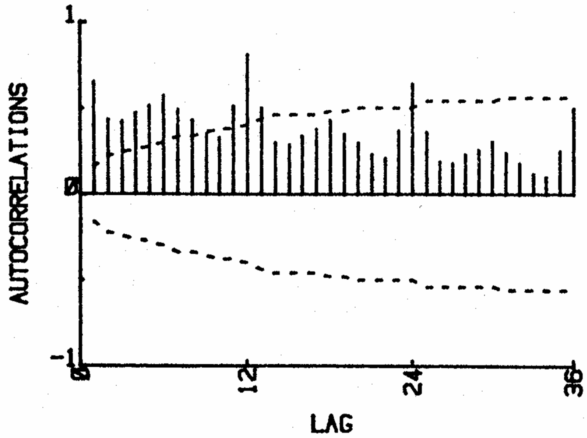


Figure 3.14 SACF OF $(1 - B)Z_t^*$

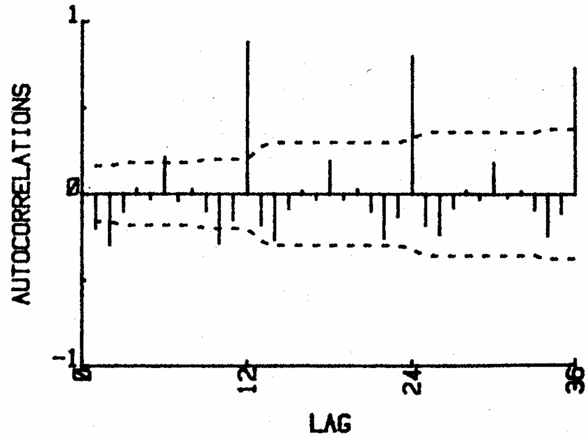


Figure 3.15 SACF OF $(1 - B^{12})Z_t^*$

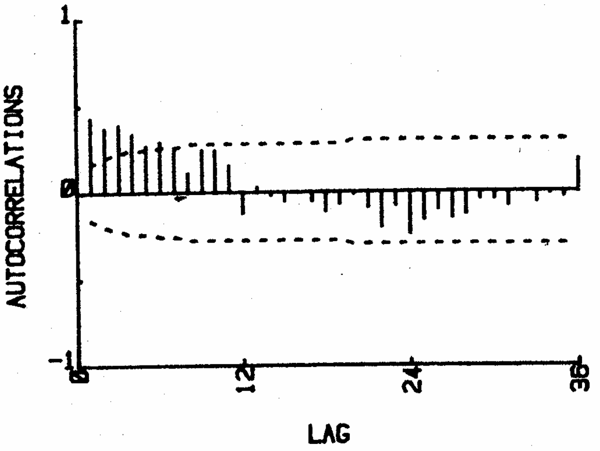


Figure 3.16 SACF OF $(1 - B)(1 - B^{12})Z_t^*$

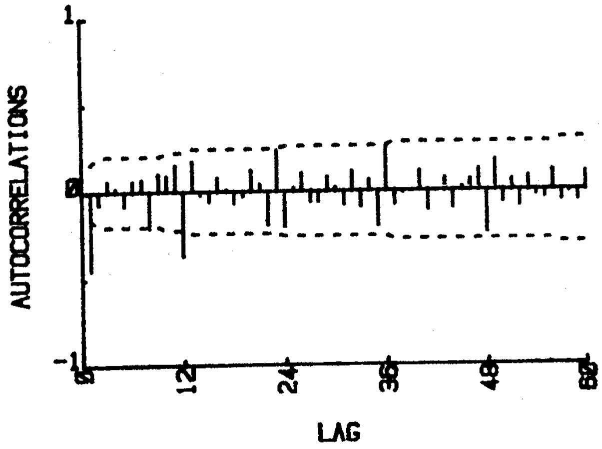


Figure 3.17 SACF OF RESIDUALS FROM EASTER AND TRADING-DAY REGRESSION

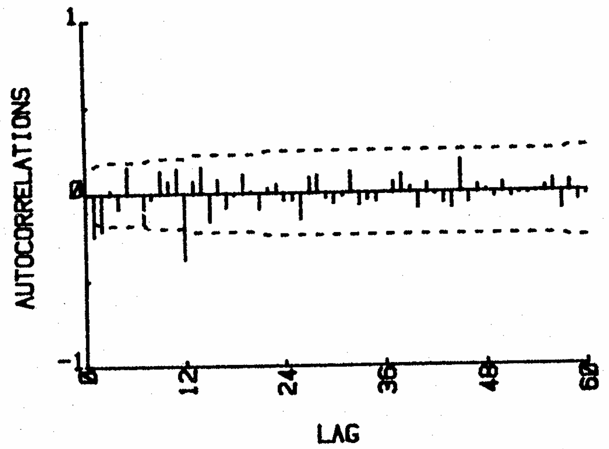


Figure 3.18 SPACF OF RESIDUALS FROM EASTER AND TRADING-DAY REGRESSION

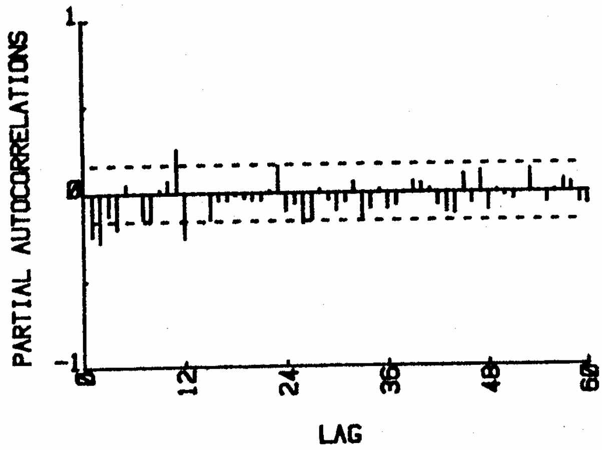


Figure 3.19 SACF OF RESIDUALS FROM MODEL (3.13) WITH $t = 9$

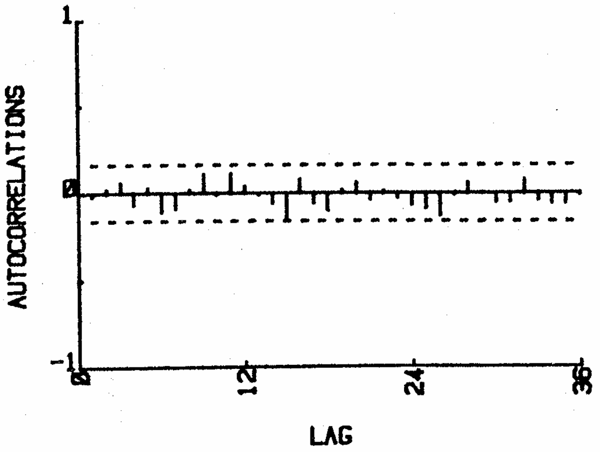


Figure 3.20 STANDARDIZED RESIDUALS FROM MODEL (3.13) WITH $t = 9$

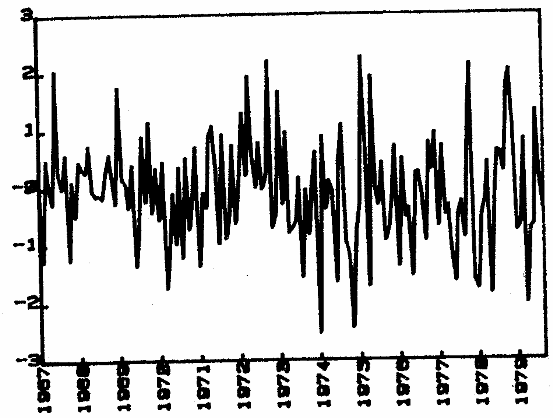


Figure 3.21a RETAIL SALES OF MEN'S AND BOYS' CLOTHING, ORIGINAL AND ADJUSTED SERIES

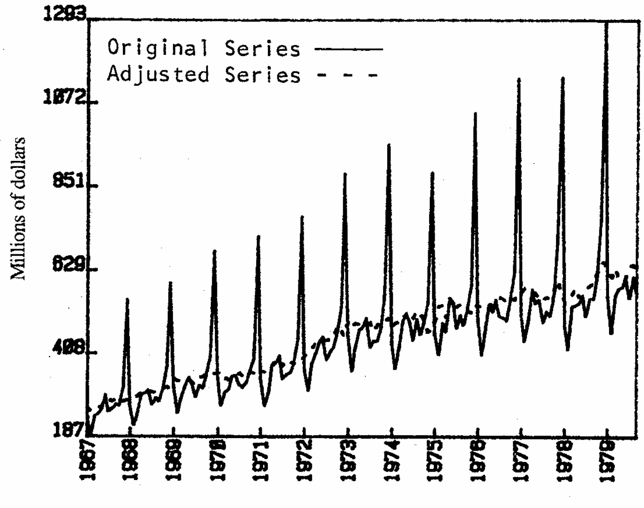


Figure 3.21b COMBINED SEASONAL, TRADING-DAY, AND EASTER FACTORS

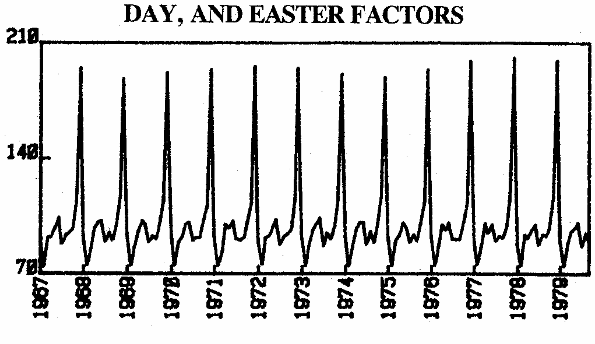


Figure 3.21c SEASONAL FACTORS

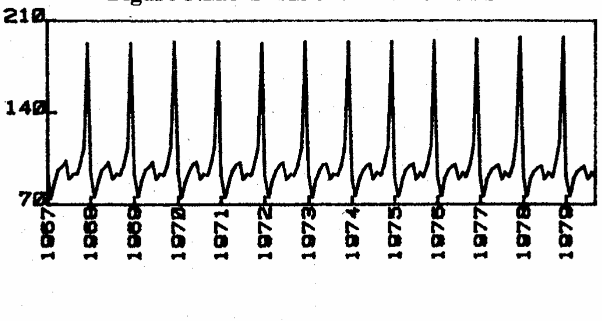


Figure 3.21d TRADING-DAY FACTORS

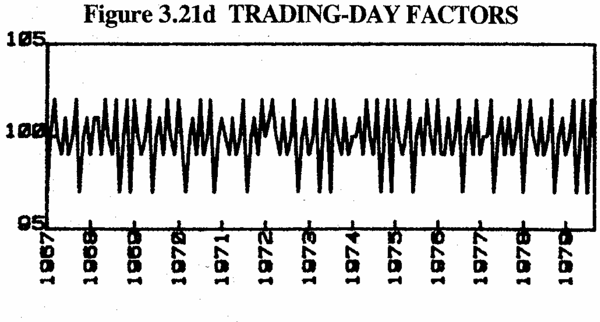
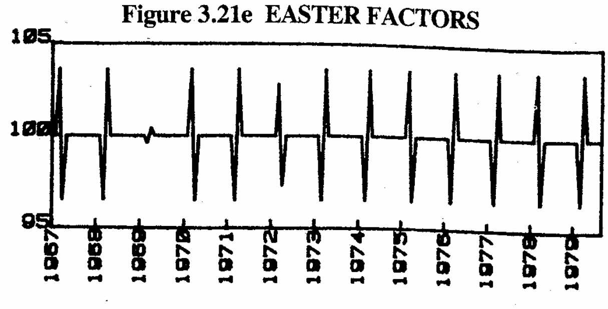


Figure 3.21e EASTER FACTORS



4.1 Additive and Innovational Outliers

We shall concentrate on two types of outliers, additive and innovational. An additive outlier (AO) is defined as

$$Z_t^* = Z_t + \omega\xi_t^{(t_0)} \tag{4.2}$$

while an innovational outlier (IO) is defined as

$$Z_t^* = Z_t + \frac{\theta(B)}{\phi(B)}\omega\xi_t^{(t_0)} \tag{4.3}$$

where Z_t follows model (2.3). In terms of the a_t's in (2.3), we have that

$$\text{(AO)}\ Z_t^* = \frac{\theta(B)}{\phi(B)}a_t + \omega\xi_t^{(t_o)} \tag{4.4}$$

and

$$\text{(IO)}\ Z_t^* = \frac{\theta(B)}{\phi(B)}\left\{a_t + \omega\xi_t^{(t_0)}\right\} \tag{4.5}$$

Thus, the AO case may be called a gross error model, since only the level of the t_0^{th} observation is affected. On the other hand, an IO represents an extraordinary shock at t_o influencing Z_{t_0}, Z_{t_0+1}, . . . through the memory of the system described by $\theta(B)/\phi(B)$.

4.2 Estimation of ω When t_0 Is Known

To motivate the procedures for the detection of AO and IO, we discuss the situation when t_0 and all time series parameters (ϕ's, θ's, and σ_a) in the model (2.3) are known. Defining the residuals $e_t = \Pi(B)Z_t^*$, where $\Pi(B) = \phi(B)/\theta(B) = (1 + \Pi_1 B + \Pi_2 B^2 + \ldots)$, we have that

$$\text{(AO)}\ e_t = \omega\Pi(B)\xi_t^{(t_0)} + a_t$$
$$\text{(IO)}\ e_t = \omega\xi_t^{(t_0)} + a_t \tag{4.6}$$

From least squares theory, estimators of the impact, ω, of the intervention and the variances of these estimators are

$$\text{(AO)}\ \hat{\omega}_A = \rho^2\Pi(F)e_{t_0},\quad \mathrm{Var}(\hat{\omega}_A) = \rho^2\sigma_a^2$$
$$\text{(IO)}\ \hat{\omega}_I = e_{t_0},\quad \mathrm{Var}(\hat{\omega}_I) = \sigma_a^2 \tag{4.7}$$

where $\rho^2 = (1 + \Pi_1^2 + \Pi_2^2 + \ldots)^{-1}$. Thus, the best estimate of the effect of an IO at time t_0 is the residual e_{t_0}, while the best estimate of the effect for an AO is a linear combination of $e_{t_0}, e_{t_0+1}, \ldots$, with weights depending on the structure of the time series model. Note that the variance of $\hat{\omega}_A$ can be much smaller than σ_a^2.

If desired, one may perform various tests among the hypotheses:

$$H_0: Z_{t_o}^* \text{ is neither an IO nor an AO}$$
$$H_1: Z_{t_o}^* \text{ is an IO}$$
$$H_2: Z_{t_o}^* \text{ is an AO}$$

The likelihood ratio test statistics for IO and AO are

$$H_1 \text{ vs } H_0 \quad \lambda_1 = \hat{\omega}_I/\sigma_a$$
$$H_2 \text{ vs } H_0 \quad \lambda_2 = \hat{\omega}_A/(\rho\sigma_a)$$

On the null hypothesis H_0, λ_1 and λ_2 are both distributed as $N(0, 1)$.

4.3 Detection of Outliers

In practice, t_0, as well as the time series parameters, are all unknown. If only t_0 is unknown, one may proceed by calculating λ_1 and λ_2 for each t, denoted by λ_{1t} and λ_{2t}, and then make decisions based on the sampling properties given above. The time series parameters (ϕ's, θ's, and σ_a) are also unknown, and it can be shown that the estimates of these parameters can be seriously biased by the existence of outliers. In particular, σ_a will tend to be overestimated. These considerations have led to the following iterative procedure to handle a situation in which there may exist an unknown number of AO or IO outliers.

(i) Model the series Z_t^* by supposing that there are no outliers (i.e., $Z_t^* = Z_t$) and from the estimated model compute the residuals

$$\hat{e}_t = \hat{\Pi}(B)Z_t^*$$

Let $\hat{\sigma}_a^2 = \frac{1}{n}\sum_{t=1}^{n} \hat{e}_t^2$ be the initial estimate of σ_a^2.

(ii) Compute $\hat{\lambda}_{it}$, $i = 1, 2$ and $t = 1, \ldots n$, these being λ_{1t} and λ_{2t} with the estimated model. Let $|\hat{\lambda}_{t_0}| = \max_t \max_i [|\hat{\lambda}_{it}|]$. If $|\hat{\lambda}_{t_o}| = |\hat{\lambda}_{1t_0}| > c$, where c is a predetermined positive constant usually taken to be 3, then there is the possibility of an IO at t_0, and the best estimate of ω is $\hat{\omega}_{It_0}$. Eliminate the effect of this possible IO by defining a new residual $\tilde{e}_{t_0} = \hat{e}_{t_0} - \hat{\omega}_{It_0} = 0$. If, on the other hand $|\hat{\lambda}_{t_0}| = |\hat{\lambda}_{2t_0}| > c$, then there is the possibility of an AO at t_0, and the best estimate of its effect is $\hat{\omega}_{At_0}$. The effect of this AO can be removed by defining the new residuals $\tilde{e}_t = \hat{e}_t - \hat{\omega}_{At_o}\hat{\Pi}(B)\xi_t^{(t_o)}$, $t \geq t_o$. A new estimate $\tilde{\sigma}_a^2$ is computed from the modified residuals.

(iii) Recompute $\hat{\lambda}_{1t}$ and $\hat{\lambda}_{2t}$ based on the same initial parameter estimates of the ϕ's and θ's but using the modified residuals and $\tilde{\sigma}_a^2$, and repeat the process (ii).

(iv) When no more outliers are found in (iii), suppose that k outliers (either IO or AO) have been tentatively identified at times $t_1, \ldots, t_k$. Treat these times as if they are known, and estimate the outlier parameters, $\omega_1, \ldots, \omega_k$ and the time series parameters simultaneously using models of the form

$$Z_t^* = \sum_{j=1}^{k} \omega_j L_j(B)\xi_t^{(t_j)} + \frac{\theta(B)}{\phi(B)}a_t \qquad (4.9)$$

where $L_j(B) = 1$ for an AO and $L_j(B) = \frac{\theta(B)}{\phi(B)}$ for an IO at $t = t_j$. The new residuals are

$$\hat{e}_t^{(1)} = \hat{\Pi}^{(1)}(B)[Z_t^* - \sum_{j=1}^{k}\hat{\omega}_j\hat{L}_j(B)\xi_t^{(t_j)}] \qquad (4.10)$$

The entire process is repeated until all outliers are identified and their effects simultaneously estimated.

The above procedure is easy to implement since very few modifications to existing software capable of dealing with ARIMA and transfer function models are needed to carry out the required computations. Based on simulation studies, the performance of this procedure for estimating the autoregressive coefficient of a simple AR(1) model compares favorably with the robust estimation procedures proposed by Denby and Martin (1970) and Martin (1980). While the latter procedures cover only the AR case, our iterative procedure can be used for any ARIMA model.

4.4 Seasonal Adjustment

For the purposes of seasonal adjustment, trading-day variation, holiday variation, and outliers can all be incorporated into the model-based procedure by writing the model as

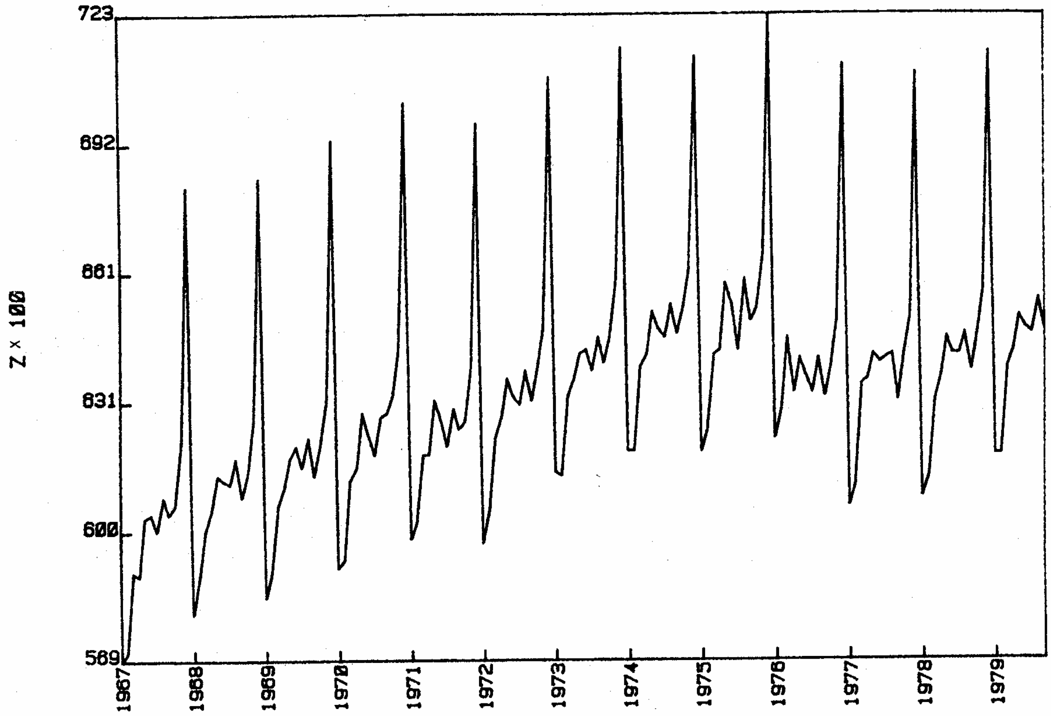
$$Z_t^* = TD_t + E_t + O_t + Z_t \qquad (4.11)$$

where $O_t = \sum_{j=1}^{k} \omega_j L_j(B)\xi_t^{(t_j)}$. The influences of all these effects can then be removed by setting $\dot{Z}_t = Z_t^* - \hat{TD}_t - \hat{E}_t - \hat{O}_t$ after which the techniques of section 2 can then be used to decompose $\dot{Z}_t$ into the canonical seasonal $\hat{S}_t$, and the canonical nonseasonal $\hat{N}_t$ (allocating appropriate parts of $\hat{TD}_t$ and $\hat{E}_t$ to $\hat{S}_t$ and $\hat{N}_t$ as discussed in sections 3.1 and 3.2). Finally, in keeping with current practice, the series adjusted for seasonal, trading-day, and holiday variation is $\hat{N}_t + \hat{O}_t$ (in the transformed metric).

4.5 An Example

To illustrate the iterative procedure described above, we consider the logarithms of the monthly retail sales of variety stores from January 1967 to September 1979 obtained from the Bureau of the Census. This series contains trading-day and

Figure 4.1 LOG RETAIL SALES OF VARIETY STORES (Z_t^*)
(Modified for trading-day and Easter effects)



Easter variation; as a preliminary step, these effects were modeled as described in section 3 and removed from the data.[2] The preprocessed data are reported in the appendix and plotted in figure 4.1.

The ARIMA model obtained from the preliminary analysis is

$$(1 - \phi_1 B - \phi_2 B^2)(1 - B)(1 - B^{12})Z_t^* = (1 - \theta_{12} B^{12})a_t \qquad (4.12)$$

where $\phi_1 = -.40$, $\phi_2 = -.27$, $\theta_{12} = .81$, and Z_t^* denotes the logarithms of the trading-day and Easter adjusted sales. Examination of the statistics $\hat{\lambda}_{1t}$ and $\hat{\lambda}_{2t}$ as described in (ii) of section 4.3 indicates the possibility of an IO at $t = 112$. After modification of the residuals, updating of the estimate of σ_a^2, and construction of new sequences for $\hat{\lambda}_{1t}$ and $\hat{\lambda}_{2t}$, an AO is identified at $t = 96$. Further inner iterations revealed two additional candidate IO's at $t = 113$ and $t = 45$.

For the second major iteration, the four tentatively identified outliers and the ARIMA parameters are simultaneously estimated. The process is repeated and an AO at $t = 121$ together with an IO at $t = 114$ are identified. The process is continued through five major iterations; table 4.1 summarizes the results. The elimination of the eight outliers leads to changes in the estimates of the time series parameters and a large reduction in the estimated variance, σ_a^2.

[2]The effects of possible outliers were approximately removed so that the trading-day and Easter parameter estimates would not be badly biased.

Table 4.1. VARIETY STORES OUTLIER DETECTION

Major iteration	$\hat{\phi}_1$	$\hat{\phi}_2$	$\hat{\theta}_{12}$	$\hat{\sigma}_a^2 \times 10^3$	Outlier time	Outlier type
(1)	−.40	−.27	.81	1.02	112	IO
					96	AO
					113	IO
					45	IO
(2)	−.53	−.27	.83	.65	121	AO
					114	IO
(3)	−.61	−.38	.85	.58	129	IO
(4)	−.64	−.36	.87	.54	103	IO
(5)	−.66	−.33	.89	.50		

Discussion. For this example, it is interesting to note that three consecutive IO's are identified at $t = 112$, 113, and 114, suggesting an intervention effect different from that of an individual AO or IO. Discussions with analysts at the Census Bureau revealed that at around $t = 112$, (which is April 1976), a major variety store chain, W. T. Grant, went out of business, and that as a result a significant proportion of retail sales previously made at variety stores were shifted to department stores. Based upon this information, it is reasonable to expect a level drop in the series starting at $t = 112$. A model for this is

$$Z_t^* = Z_t + \frac{\omega}{1 - B}\xi_t^{(112)} \text{ with } \omega < 0 \qquad (4.13)$$

The important point is that the outlier analysis summarized in table 4.1 is consistent with the above explanation, and a closer examination of consecutive outliers may reveal the nature of the intervention. To illustrate, the standardized residuals from the preliminary model for $t = 111, \ldots, 115$ are reported below.

t	$\hat{e}_t/\hat{\sigma}_a$
111	0.0
112	−5.3
113	−3.1
114	−2.1
115	−.2

If the model for Z_t^* is described by (4.13) with Z_t following the preliminary model (4.12), then the residuals are

$$e_t = a_t + \frac{(1 - \phi_1 B - \phi_2 B^2)(1 - B^{12})}{(1 - \theta_{12} B^{12})} \omega \xi_t^{(112)} \quad (4.14)$$

From the initial estimate of θ_{12} in table 4.1, it follows that $(1 - B^{12})/(1 - \theta_{12} B^{12})$ is approximately 1. From 4.14 and the initial estimates of ϕ_1 and ϕ_2, we would expect to observe large negative residual values at $t = 112$, 113, and 114 that would behave somewhat like ω, $.4\omega$, and $.3\omega$. Also, there would be no effect on the residual at $t = 115$. The three consecutive IO's noted above are thus reasonably consistent with the model (4.13).

If we consider the first differences of the data, W_t^*, $= Z_t^* - Z_{t-1}^*$, then the model (4.13) reduces to

$$W_t^* = W_t + \omega \xi_t^{(112)}$$

where $W_t = Z_t - Z_{t-1}$. In other words, the intervention in (4.13) can be viewed as an AO at $t = 112$ for the differenced data W_t^*. This is confirmed by applying the outlier detection procedure to W_t^* resulting in the identification of a single AO at $t = 112$ instead of three consecutive outliers starting at that point.

5. REVISIONS

Most seasonal adjustment methods recently proposed or currently in use use two-sided filters. When current data are being adjusted, the future values of the series required for the use of the two-sided filters are not available. In practice, the estimates of N_t are computed and subsequently changed as more data become available—these changes are known as revisions. Revisions present a practical problem, since it can be difficult to explain to users of seasonally adjusted data why the current adjusted data get changed in subsequent years—especially if the changes are large.

Large revisions are often viewed as undesirable. However, it is easy to devise a seasonal adjustment method which produces zero revisions simply by using one-sided filters. Since such methods have rarely been adopted in practice, we must conclude that zero revisions, as well as large revisions, are undesirable. From a signal extraction point of view, estimates of N_t should be updated as new data become available, and the magnitude of the revisions should depend on the model and the data. For the above reasons, we do not feel one method of seasonal adjustment should be preferred over another on the basis of revisions. However, because of the emphasis that has been placed on revisions and their practical significance, it is of interest to investigate how the model-based method behaves in comparison with other methods in terms of revisions. In section 5.1 we make some theoretical comparisons between revisions for the model-based method and those for X−11, and in section 5.2 we report on results of an empirical study comparing the revisions for the model-based method, X−11, and X−11 ARIMA.

5.1 Theoretical Comparisons

To make some theoretical comparisons of revisions for the proposed model-based method with those of X−11, we consider the following simplified situation:

(i) Z_t follows the model

$$(1 - B)(1 - B^{12})Z_t = (1 - \theta_1 B)(1 - \theta_{12} B^{12})a_t \quad (5.1)$$

(ii) the standard options of the additive version of X−11 are used, and

(iii) at the end of the series the values needed to apply the symmetric X−11 filters are obtained by forecasting future observations based on (5.1).[3]

Seasonal adjustment is usually done at the Census Bureau by producing in December forecasts of the seasonal components for the coming year and using these to adjust the data as they become available. We refer to this practice as year-ahead seasonal adjustment. To duplicate this procedure for both X−11 and the model-based method, we assume that (i) the respective symmetric filters are applied to the series extended with forecasts of Z_t to produce forecasts of the seasonal components for the 12 months beyond the end of the observed series and (ii) the forecasted seasonal components are used to adjust the data as they become available. Note that for the model-based procedure, these forecasted seasonal components are the conditional expectations (under normality) of future S_t's, given the available observations.

For theoretical convenience, we shall measure the magnitude of revisions by their mean square. If $A_t^{(0)}$ is the initial adjusted value (estimate of N_t) and $A_t^{(1)}$ the revised adjusted value when 1 year of additional observations is available, then $A_t^{(1)} - A_t^{(0)}$ is the first-year revision. To compare the model-based procedure with X−11 we shall consider the ratio of $E[(A_t^{(1)} - A_t^{(0)})^2]$ for model-based to that for X−11. Note that the seasonal components used in the computation of $A_t^{(0)}$ have been forecasted from 1 to 12 months ahead. To reduce the number of comparisons, we shall only consider the cases where the seasonal components are forecast 1, 6, and 12 months ahead.

[3]This is not how X−11 is actually used, and will give smaller mean squared revisions than the way X−11 actually operates (by results of Pierce 1980 and Geweke 1978).

Under our simplifying assumptions, the results in Pierce (1980) can be used to obtain the mean squared first-year revisions for the model-based approach and for X−11. The ratio of the mean squared revisions for a grid of θ_1 and θ_{12} values is presented in table 5.1.

Table 5.1. RATIO OF MEAN SQUARED FIRST-YEAR REVISIONS (MODEL-BASED/X−11)

1-month-ahead forecasts

		θ_1 .1	.3	.5	.7	.9
	.1	1.56	1.38	1.19	1.19	1.28
	.3	1.35	1.22	1.08	1.08	1.14
θ_{12}	.5	1.06	.98	.89	.89	.93
	.7	.66	.62	.58	.58	.60
	.9	.15	.15	.15	.14	.15

6-month-ahead forecasts

		θ_1 .1	.3	.5	.7	.9
	.1	1.48	1.48	1.48	1.48	1.48
	.3	1.27	1.27	1.27	1.27	1.27
θ_{12}	.5	1.00	1.00	1.00	1.00	1.00
	.7	.62	.62	.62	.62	.62
	.9	.15	.15	.15	.15	.15

12-month-ahead forecasts

		θ_1 .1	.3	.5	.7	.9
	.1	1.28	1.31	1.36	1.41	1.45
	.3	1.14	1.16	1.19	1.23	1.25
θ_{12}	.5	.94	.95	.96	.98	.99
	.7	.63	.63	.63	.63	.62
	.9	.18	.17	.16	.16	.15

The main conclusion that can be drawn from this table is that the mean squared revisions for the model-based approach are smaller than those for X−11 when θ_{12} is greater than about .4, and vice versa when θ_{12} is less than about .4.

If Z_t follows an ARIMA model, then the weight function used to compute the estimates of the nonseasonal component is

$$W_N(B) = \frac{\sigma_c^2\eta_N(B)\eta_N(F)\phi_s(B)\phi_s(F)}{\sigma_a^2\theta(B)\theta(F)} = \frac{\gamma(B)\gamma(F)}{\theta(B)\theta(F)}$$

where $\gamma(B)$ is a polynomial in B of finite degree and $\theta(B)$ is the moving average polynomial of the model (2.3). In the important case where $\theta(B) = \theta^*(B)(1 - \theta_{12}B^{12})$ with $\theta^*(B)$ a nonseasonal polynomial in B, the magnitude of θ_{12} is the most important factor in determining the effective length of the moving average filter $W_N(B)$. Larger values of θ_{12} lead to longer moving averages. Cleveland and Tiao (1976) found that an implicit model for the additive version of X−11 with standard options can be roughly regarded as having $\theta(B) = \theta^*(B)(1 - \theta_{12}B^{12})$ with θ_{12} about .4. Thus, when θ_{12} is less than about .4, the model-based method uses shorter filters than X−11 and otherwise uses longer filters. Since .4 is the approximate value for θ_{12} in table 5.1 at which the mean squared revisions for the model-based method become smaller than those for X−11, this suggests that longer filters lead to smaller first-year revisions. Because the key factor here is the size of θ_{12}, results similar to those in table 5.1 might be expected for models other than (5.1) with $\theta(B) = \theta^*(B)(1 - \theta_{12}B^{12})$.

The above results do not mean that the X−11 procedure should be preferred over the model-based procedure whenever θ_{12} is estimated to be less than about .4 and vice versa. In our opinion, justifications for the use of the model-based procedure are based on grounds other than the comparative size of the revisions. In fact, these results cast severe doubt on the appropriateness of using revision measures to choose between different methods of seasonal adjustment. The parameter θ_{12} can be estimated from the data, so there is information in the data regarding how long a filter is appropriate. It makes sense to use this information as the model-based method does rather than ignore it as use of X−11 with standard options does. The theoretical results together with the empirical findings in the next section are presented in this paper mainly because of the practical interest in this problem.

5.2 Empirical Comparisons

Measures of revisions. In this section, we report on the results of an earlier empirical study comparing revisions for the model-based approach, X−11, and X−11 ARIMA. (See Hillmer 1981.) In this study, the magnitude of revisions was assessed by measures other than the mean square measure used in the preceding discussion, where it was chosen for theoretical convenience. We would expect the results based upon different measures to yield broadly similar conclusions.

For any particular 12-month period, let $A_t^{(0)}$ $t = 1, \ldots, 12$ denote the year-ahead adjusted values, and $A_t^{(k)}$ $t = 1, \ldots, 12$ and $k = 1, 2, 3$ denote new adjusted values for the same 12-month period after k additional years of data become available. The revisions in level after k years are then $R_t^{(k)} = A_t^{(k)} - A_t^{(0)}$ $t = 1, \ldots, 12$ and $k = 1, 2, 3$. To measure the magnitude of the revisions in the level of the series, we can compute the average absolute revision over a 12-month period after k years of additional data are available

$$D^{(k)} = \frac{1}{12}\sum_{t=1}^{12}|R_t^{(k)}| \qquad k = 1, 2, 3 \tag{5.2}$$

Another quantity of interest is the magnitude of the revisions in the month-to-month percentage changes. We let $P_t^{(0)} = 100(A_t^{(0)} - A_{t-1}^{(0)})/A_{t-1}^{(0)}$ $t = 2, \ldots, 12$ denote the month-to-month percentage changes in the year-ahead adjusted figures, and let $P_t^{(k)} = 100(A_t^{(k)} - A_{t-1}^{(k)})/A_{t-1}^{(k)}$ $t = 2, \ldots, 12$ be the month-to-month percentage changes after k years of additional data are available. Then,

$$C^{(k)} = \frac{1}{11}\sum_{t=2}^{12}|P_t^{(k)} - P_t^{(0)}| \qquad k = 1, 2, 3 \tag{5.3}$$

is a measure of the revision in the month-to-month percentage changes.

The data used. To perform the empirical comparison, actual time series were obtained from the Census Bureau and the Bureau of Labor Statistics. The series included components of the following groups: Employment-unemployment series, industrial inventories, wholesale inventories, construction series, retail sales, wholesale sales, industrial shipments, and retail service series. The series were not a random sample but were not chosen with any particular characteristics in mind.

To perform the ARIMA model-based adjustment, each series was modeled in a manner similar to that given in Box and Jenkins (1970) and in section 3. The research on the modeling of outliers as described in section 4 was completed subsequent to the empirical study, so the treatment of outliers in the study was slightly different. In particular, all outliers were treated as additive outliers. A few series were eliminated from the study at the modeling stage for reasons including (i) the apparent lack of the need for the seasonal differencing operator $1 - B^{12}$, indicating that a seasonal component should not be removed (see sec. 2.4) and (ii) modeling complications such as the occurrence of a strike in the series. We plan on further investigating the series removed for the second reason. After eliminating these series, 76 series were available for the study. Each of the 76 series was seasonally adjusted by the three methods under consideration. For the model-based method, logarithms of the data were taken for 58 of the series, square roots for 2 series, cube roots for 1 series, and 15 series were not transformed at all. For X−11 and X−11 ARIMA, the additive versions with standard options were applied to the 15 series not transformed for the model-based method, and to the logarithms of the data for all the other series.[4] The adjusted transformed series were transformed back to the original metric by exponentiating, squaring, or cubing before computing the revision measures.

For each method, the seasonal adjustment was performed using data from the beginning of the series up to 3 years from the end of the series, and year-ahead adjusted values were computed. Then 1, 2, and 3 years of data were sequentially added so that first-, second-, and third-year measures of (5.2) and (5.3) could be computed. For seven of the series, only first-year measures were calculated because these series were felt to be too short for the computation of the second- and third-year revisons.

Comparison of model-based and X−11 methods. In comparing the model-based approach to X−11 for each of the two measures (5.2) and(5.3), the ratios of the revision measures (model-based divided by X−11) were computed.[5] Histograms of these ratios for the first-, second-, and third-year revisions are plotted on a log scale in figures 5.1 and 5.2 for the measures (5.2) and (5.3). The geometric means of these ratios are reported in table 5.2. Based upon the figures and the table, we draw the following conclusions.

(i) On the average, there was at least a 40-percent reduction in the first-, second-, and third-year revisions in the level of the series when the model-based procedure was used. From the histograms, there appears to be a moderate amount of variation in performance; but, for the majority of the series considered there was a reduction in the revisions in level when the model-based method was used.

(ii) There also was more than a 40-percent average reduction in the revisions of month-to-month percentage changes for the first, second, and third years for the model-based approach. Again, a major proportion of the series had smaller revisions in month-to-month percentage changes when the model-based approach was used rather than X−11.

Table 5.2. GEOMETRIC MEANS OF THE RATIO OF MODEL-BASED TO X−11

Revisions in the level of the series		
First year	Second year	Third year
.59	.56	.56
Revisions in month-to-month percentage changes		
First year	Second year	Third year
.56	.54	.54

Comparison of model-based and X−11 ARIMA methods. For comparing the model-based approach with X−11 ARIMA, similar ratios of the revision measures (model-based divided by X−11 ARIMA) were computed. Histograms of these ratios, plotted in log scale, are presented in figures 5.3 and 5.4 for the measures (5.2) and (5.3), and the geometric means of these ratios are reported in table 5.3. From these, we draw the following conclusions.

(i) For each of the first-, second-, and third-year revisions, there was more than a 40-percent average reduction in the revisions in level when the model-based approach was used instead of X−11 ARIMA.

(ii) For revisions in the month-to-month percentage changes, the use of the model-based approach led to a reduction of more than 40 percent in each of the first-, second-, and third-year figures on average.

Table 5.3. GEOMETRIC MEANS OF THE RATIO OF MODEL-BASED TO X−11 ARIMA

Revisions in the level of the series		
First year	Second year	Third year
.53	.55	.59
Revisions in month-to-month percentage changes		
First year	Second year	Third year
.58	.56	.58

[4]For X−11 ARIMA, no account was taken of trading-day variation in forecasting the observed series (it was handled in the usual X−11 way in adjusting the extended series) because the X−11 ARIMA computer program did not allow for it. Smaller revisions for X−11 ARIMA may have resulted if some allowance for trading-day effects had been made in the forecasting.

[5]The ratios were computed to make comparisons across the 76 series. The revisions for the individual series for each of the three methods are available and will be provided on request.

Figure 5.1 RATIOS (MODEL-BASED/X-11) FOR REVISIONS IN LEVEL

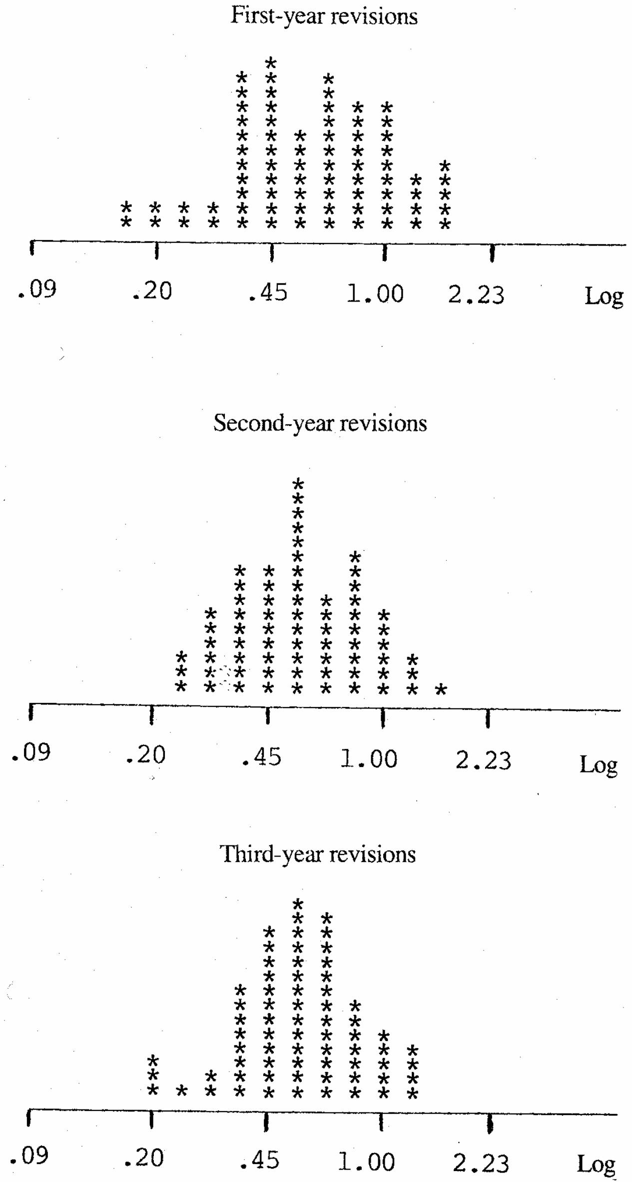


Figure 5.2 RATIOS (MODEL-BASED/X-11) FOR REVISIONS IN MONTH-TO-MONTH PERCENTAGE CHANGES

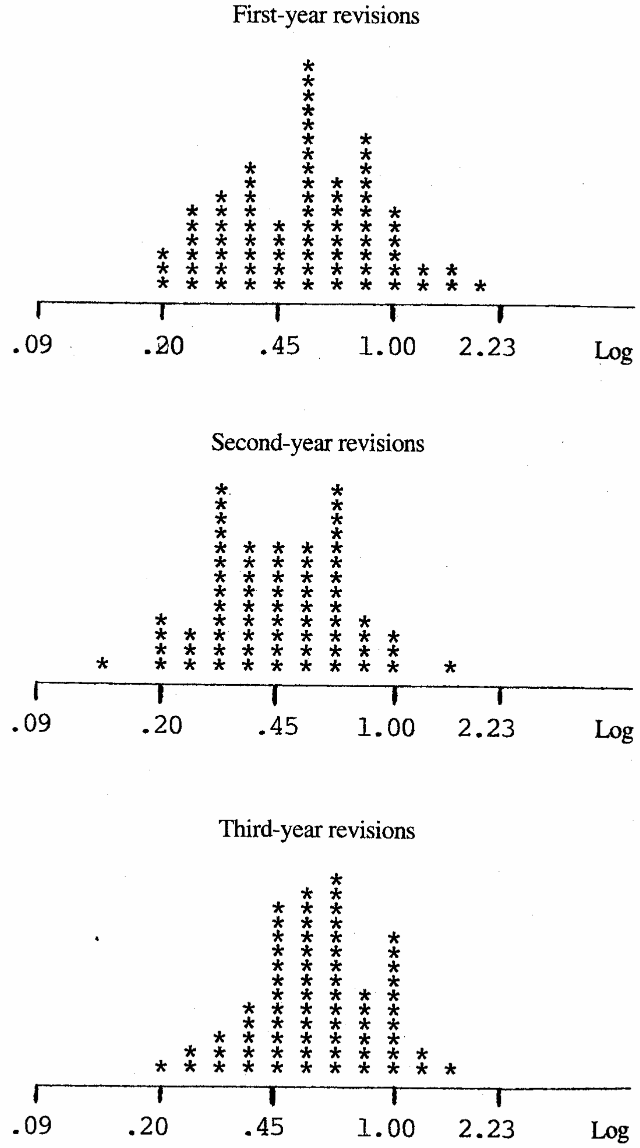


Figure 5.3 RATIOS (MODEL-BASED/X-11 ARIMA) FOR REVISIONS IN LEVEL

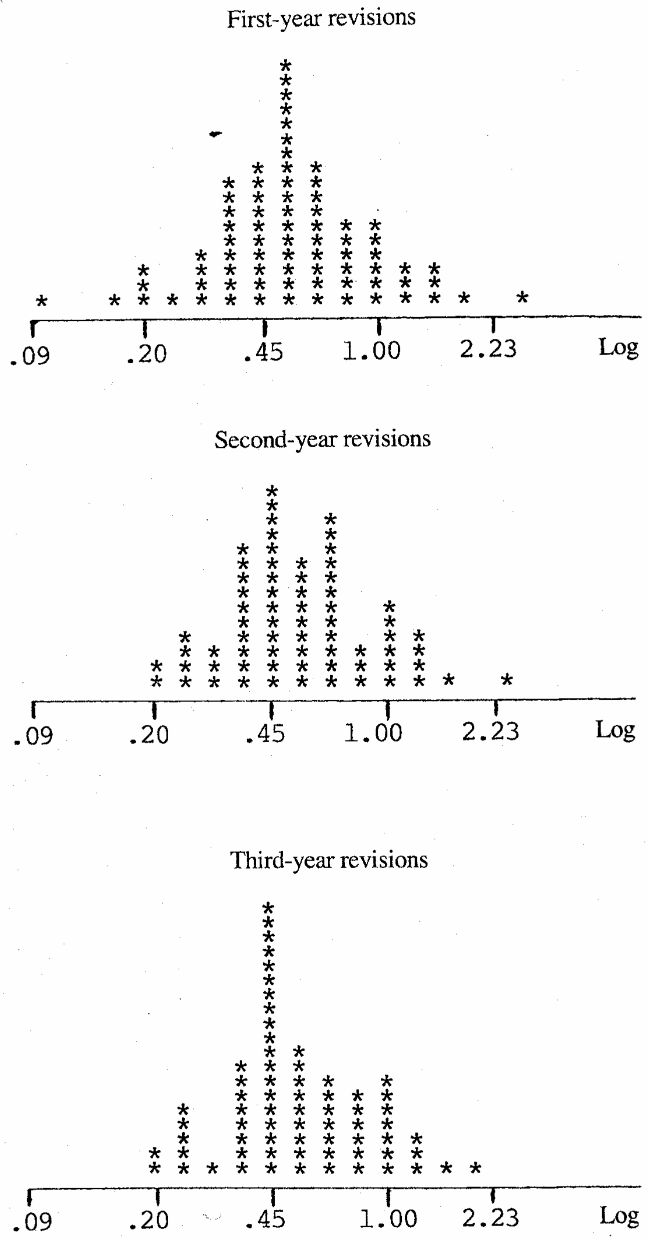
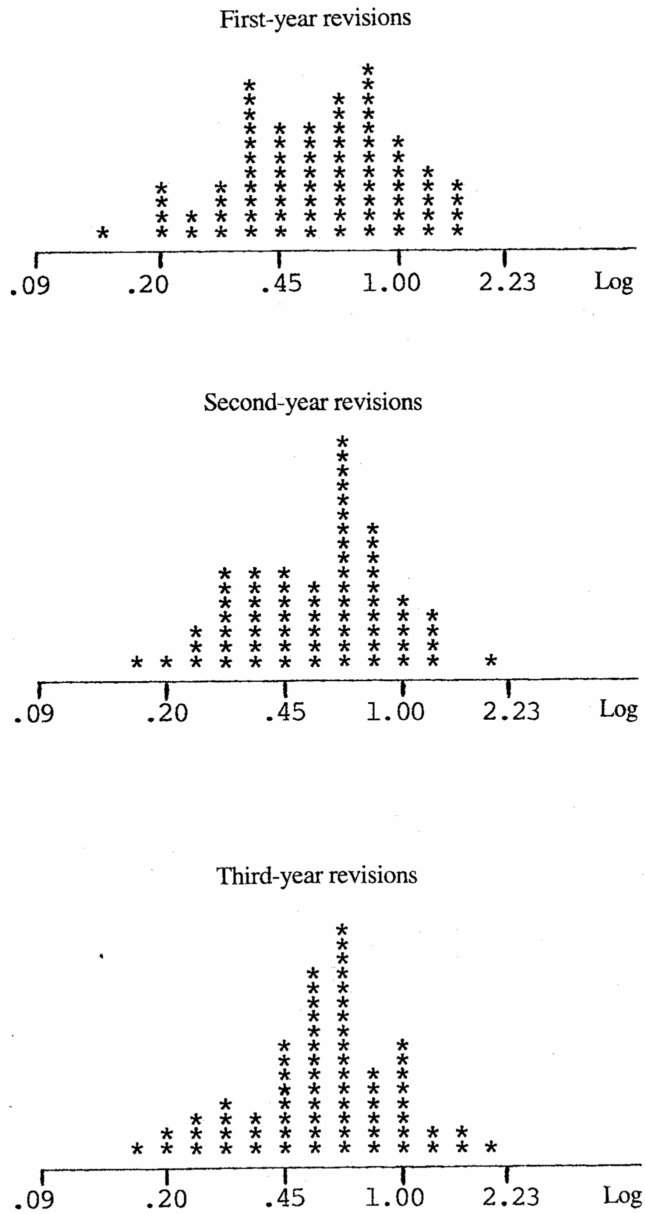


Figure 5.4 RATIOS (MODEL-BASED/X-11 ARIMA) FOR REVISIONS IN MONTH-TO-MONTH PERCENTAGE CHANGES



Explanation of results. It is of interest to investigate why we obtained substantially smaller revisions when using the model-based approach rather than the other two methods. For the 76 series studied, the moving average part of the model for the series Z_t has the form $\theta(B) = \theta^*(B)(1 - \theta_{12}B^{12})$ where $\theta^*(B)$ is a nonseasonal moving average polynomial. The estimated values of θ_{12} for the 76 series are reported in table 5.4. Notice that about 99 percent of the series had $\hat{\theta}_{12}$ values larger than .4. Thus, from the results in section 5.2, we would expect the model-based approach to have smaller revisions than X−11 with standard options since it is using longer filters. The situation seems to be similar when comparing model-based to X−11 ARIMA with standard options. To the extent that series to be adjusted follow models with high θ_{12} values, as in this study, we can expect the model-based approach to lead to smaller revisions than X−11 or X−11 ARIMA.

Table 5.4 ESTIMATED VALUES OF θ_{12} FOR THE SERIES IN THE STUDY

	$\hat{\theta}_{12}$					
Occurrences	< .5	.5-.6	.6-.7	.7-.8	.8-.9	.9-1.0
Number	1	5	3	11	38	18
Percent	1.3	6.6	4.0	14.5	50.0	23.7

6. COMMENTS ON MODEL-BASED VERSUS EMPIRICAL PROCEDURES

We believe that the best way to make progress in the area of seasonal adjustment is through an iteration between theory and practice. In this spirit, we recognize that empirical seasonal adjustment procedures, particularly X−11, have served a need for many years, and have provided a starting point in the development of model-based procedures. On the other hand, if we are going to make progress in seasonal adjustment, it is necessary that model-based procedures be developed. We now make some general comments on the relative merits of empirical and model-based approaches.

Underlying statistical assumptions. In model-based procedures, the underlying statistical assumptions are specified so that methods can be constructively criticized from a statistical viewpoint. Thus, it is possible to understand the statistical principles implicit in model-based methods. Also, when valid problems arise, a model-based approach will provide possible ways for improvement.

In contrast, empirical adjustment methods are not based upon statistical theory, so that it is difficult or impossible to judge them on theoretical grounds. This makes improvements difficult with empirical methods. Therefore, we expect that more rapid progress in improving seasonal adjustment methods should be possible with model-based methods rather than empirical methods.

Arbitrariness. As discussed in section 2, seasonal adjustment is inherently arbitrary. The ARIMA model-based approach presented here assumes that an appropriate model can be found for the observed series. Modeling time series is a somewhat arbitrary process, but any model finally used has been subjected to appropriate diagnostic tests. The primary source of arbitrariness lies in the fact that even if the model for Z_t is known, the proposed approach depends upon some arbitrary principles to achieve a unique decomposition. The advantage here is that the arbitrariness is clearly specified. Thus, if someone disagrees with the choice of the canonical decomposition, they may choose a different acceptable decomposition, or if they are unwilling to make a choice, they may refrain from seasonal adjustment.

If we view seasonal adjustment as a signal extraction problem (as most empirical adjustment procedures implicitly do), then it is clear that empirical methods must somehow deal with the same kind of arbitrary choices as the ARIMA model-based approach. The problem is that with empirical methods the nature of the arbitrary choices being made is unclear. As a result, there is no way to judge the reasonableness of the choices. In fact, as discussed subsequently, empirical procedures need not be consistent with the information in the data; consequently, some of the implicit choices may be, in a sense, incorrect.

Consistency with the data and flexibility. The model-based seasonal adjustment procedure described in section 2 uses models for S_t and N_t that are constrained to satisfy (2.4). Therefore, given that one has built an appropriate model for Z_t, the estimates derived under this procedure are consistent with the information in the data. The model-based approach is also flexible because for each different series a model of appropriate form can be selected and the model parameters estimated using the relevant data. Model identification is a somewhat subjective procedure that can, nevertheless, be checked, while model estimation is totally objective.

In contrast, an empirical seasonal adjustment method will be consistent with the information in the data only by chance. For example, the additive X−11 program with standard options could be thought of as being consistent with the data if Z_t approximately follows the ARIMA model given by Cleveland and Tiao (1976). If the data deviate substantially from this model, then the additive X−11 will be inconsistent with the information in the data. The X−11 program does have many options that can be adjusted by analysts. The frequency with which nonstandard options are selected varies considerably, depending on who is adjusting the data. When this is done it is frequently because previously estimated seasonal factors appear to be inadequate. These options offer only limited flexibility, so it may be that no choice of options yields a procedure that is consistent with the data. Also, choosing options is highly subjective, requiring judgment and experience, and without valid statistical tests to support one's judgment, inappropriate options may be selected.

Optimality of filters. In empirical approaches, the moving averages have been determined by trial and error, and therefore do not necessarily satisfy any particular optimality criterion. In contrast, the moving averages for model-based approaches are derived from the theory of signal extraction so that given the models for S_t and N_t the estimates have smaller mean squared error than any other linear unbiased estimates. Finally note that, in principle, model-based approaches could derive estimates that satisfy a criterion other than minimum mean squared error. (See Wecker 1979, for example.)

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APPENDIX

DATA SETS USED IN THE EXAMPLES

EMPLOYED MALES IN NONAGRICULTURAL INDUSTRIES: 1/65-8/79

(Thousands)

1963	2071	2057	2228	2354	2777	3449	3415	2510	2549	2549	2622
2482	2489	2470	2636	2659	3434	4009	3910	2699	2680	2686	2688
2520	2533	2558	2607	2708	3400	3788	3717	2615	2605	2505	2556
2350	2449	2520	2652	2784	3407	3868	3897	2720	2786	2736	2793
2680	2700	2742	2880	2850	3570	4059	3970	2941	2944	2922	2974
2774	2786	2847	2815	2939	3428	3991	3827	2834	2833	2868	2928
2713	2758	2712	2838	2963	3451	4118	3933	2918	2997	3066	3049
2932	2880	2987	3144	3275	3937	4336	4230	3258	3325	3306	3333
3109	3236	3388	3452	3474	4227	4634	4426	3568	3634	3535	3534
3406	3345	3422	3444	3586	4242	4661	4429	3484	3546	3414	3383
3124	3140	3134	3205	3267	3833	4282	4138	3216	3276	3242	3237
3104	3111	3176	3331	3439	4034	4576	4342	3300	3422	3330	3401
3231	3253	3330	3512	3683	4312	4791	4586	3564	3778	3715	3764
3566	3472	3520	3679	3842	4582	4941	4798	3743	3826	3792	3745
3637	3501	3634	3704	3787	4528	4936	4586				

WHOLESALE SALES OF HARDWARE: 1/67-11/79

(Millions of dollars)

626	614	689	686	723	778	711	824	793	831	775	689
692	718	757	769	791	809	836	878	856	935	850	763
761	796	830	902	910	932	931	908	934	995	865	822
763	778	841	845	863	952	909	899	952	963	893	831
773	803	918	967	963	1065	1014	1051	1054	1051	1039	960
930	956	1072	1023	1136	1181	1088	1247	1164	1251	1218	1062
1114	1088	1253	1254	1354	1349	1305	1420	1313	1481	1387	1284
1310	1262	1446	1573	1634	1612	1591	1640	1590	1696	1456	1296
1311	1232	1274	1388	1374	1443	1466	1454	1538	1587	1406	1341
1351	1367	1553	1588	1591	1703	1643	1711	1731	1678	1678	1580
1515	1544	1817	1838	1925	2017	1898	2068	1961	2027	1974	1820
1790	1708	2021	2102	2306	2360	2247	2412	2159	2455	2250	2057
2142	1984	2319	2374	2592	2461	2524	2678	2399	2794	2415	

RETAIL SALES OF MEN'S AND BOYS' CLOTHING STORES: 1/67-9/79

(Millions of dollars)

237	187	241	245	259	296	252	260	271	267	320	549
266	216	252	297	302	310	270	288	280	316	372	594
319	249	287	320	342	329	291	321	315	361	400	680
338	268	304	313	348	350	321	317	333	364	396	719
336	267	303	375	382	401	341	351	357	382	447	771
364	310	379	408	439	451	390	413	424	469	534	884
452	361	426	470	477	502	424	442	442	479	562	961
437	368	427	495	514	492	443	500	458	492	542	889
459	403	490	467	556	542	474	510	483	527	591	1044
495	404	463	540	518	552	505	502	496	558	629	1137
511	440	496	578	542	550	492	518	507	569	708	1141
480	421	532	536	542	563	508	554	552	609	763	1293
561	462	564	582	586	615	553	612	570			

RETAIL SALES OF VARIETY STORES: 1/67-9/79
(Millions of dollars. Modified for trading-day and Easter effects)

296	303	365	363	417	421	404	436	421	429	499	915
331	361	402	426	460	457	451	476	436	464	525	939
345	364	427	445	478	492	469	501	459	494	548	1022
370	378	453	470	534	510	485	527	536	553	621	1122
394	411	482	484	550	525	494	537	513	521	596	1069
393	425	503	529	581	558	547	588	549	593	649	1191
463	459	554	576	615	619	589	637	601	642	737	1279
490	490	598	615	681	654	637	694	645	684	749	1245
489	511	612	623	726	692	623	734	662	684	781	1386
503	537	636	560	607	585	559	608	556	596	665	1229
427	450	573	579	615	601	608	617	550	616	673	1199
438	458	548	584	639	616	614	647	588	648	713	1261
483	483	593	620	672	650	643	702	654			

COMMENTS ON "MODELING CONSIDERATIONS IN SEASONAL ADJUSTMENT OF ECONOMIC TIME SERIES" BY S. C. HILLMER, W. R. BELL AND G. C. TIAO

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1. INTRODUCTION

This is a very constructive and valuable contribution, which fills out the practical aspects of the model-based signal extraction approach. The essential facilities that are needed by users comprise trading-day adjustment, the Easter effect, and identification and modification of outliers. The authors conclude with some comparisons of the size of revisions for signal extraction, X−11, and X−11 ARIMA.

We have been developing a program for signal extraction in the Bank of England over the past 3 years, which is basically the same as in this paper, but there are some important divergences, which I shall discuss.

2. MODEL ESTIMATION AND DECOMPOSITION

The authors confine the decomposition of the spectrum to its seasonal and nonseasonal components. We have divided the spectrum into three parts, separating trend from irregular, to identify outliers, which the authors handle in a quite different way. For this purpose, the decomposition by partial fractions does not always produce an acceptable division between trend and irregular; for example, if the numerator of the rational lag function defining the model is of lower degree than the denominator ("bottom-heavy"), a white noise irregular leaves quite a lot of power in the high frequency part of the trend; in the time domain, this shows up as a high weight being given to the central observation in the trend filter. In Burman (1980) a smoother trend was obtained by allowing the irregular component to be a first-order moving average. On the other hand, top-heavy models (degree of numerator greater than that of denominator) sometimes generate nonmonotonic trend spectra (with a maximum instead of a minimum at frequency π). This could also occur for more complex balanced models; for example, the (0, 2, 2) nonseasonal operator with $1 + \theta_1^2 + \theta_2^2 + 6\theta_2 < 0$ (which implies that the roots of the MA operator are complex). By extending the use of a first-order moving average irregular component to these cases, a monotonic decreasing trend could be obtained (e.g., by subtracting a function of the form $(e_0 - e_1 \cos \omega)$ from the trend spectrum where $e_0 > 0, e_1 > 0$). In practice, we find that the attempt to make the trend spectrum monotonic may lead to an invalid irregular spectrum—one that takes negative values.

Hillmer, Bell, and Tiao (HBT) confine the seasonal operator to $(0, 1, 1)_s$, a restriction that Statistics Canada has imposed in the light of experience (Dagum 1979). A few series are known for which the $(0, 1, 2)_s$ operator fits better, but this does not seem to provide valid spectral decompositions. Another possibility that occurs occasionally is $(1, 0, 0)_s$ with $\phi_{12} > 0$. This is rejected in section 2.4 of the paper on intuitive grounds.

Before considering the special features developed by the authors, I have a question about the method of estimation of the filters (2.5). A very large number of forecasts and backcasts are required before the filter weights die away if θ_{12} is close to 1. Did they use what was called in Burman (1980) the Tunnicliffe Wilson algorithm? This enables exact results to be obtained with only 2-years' forecasts and backcasts and a couple of matrix inversions.

3. TRADING-DAY ADJUSTMENT AND EASTER

The model of trading-day adjustments (TDA) is well-established as part of X−11, and it is clearly defined. It therefore seems right that for model-based seasonal adjustment, it should be estimated simultaneously with the ARIMA model.

The definition of an Easter effect is one that I have not met before, but it seems eminently reasonable and simple to estimate in practice. The procedure parallels that with TDA, except that the estimate of τ, the length of the Easter bulge, is obtained by a grid search. However, the model is less clearly defined because τ is not very well-determined; so it might be advisable to keep τ constant when updating the model each year to minimize revisions.

4. OUTLIERS

4.1 Comparison of Methods

The older methods of seasonal adjustment deal with outliers in a largely intuitive way. After preliminary seasonal adjustment, outliers are identified, the series is modified, and the main procedure repeated. An outlier is identified at time t when $|I_t|$ is larger than some multiple of the RMS of the I_t (in the additive case), say $2.5\sigma_I$. In X−11 the modification is simply to subtract I_t from z_t. The Bank of England's moving average program (see Burman 1965) takes account of the weight d_0 of the outlier in the estimated trend, and the modification is I_t/w_0, where $w_0 = 1 - d_0$. This program (called hereafter B/E(1965)) also allows for the interaction among adjacent outliers of the same sign. In both programs, tapered modifications are made for smaller values of $|I_t|$, which diminishes the effect on the seasonal adjustments of an observation crossing the boundary at $2.5\sigma_I$, when the annual update takes place. Hitherto our signal-extraction program has followed this method for handling outliers.

We need to rethink the handling of outliers when using model-based methods, and HBT have now put it on a firm statistical basis. The paper draws an important distinction between additive outliers (AO) and innovative outliers (IO). An AO has only a transient effect on the series; while an IO results in an abrupt permanent change in the level, growth rate, or seasonal pattern of the series, or in all three. A more fundamental classification is between outliers that can be attributed to external causes and those that cannot. I believe that all IO's will have external causes, as will some AO's (e.g., those related to strikes or exceptional weather). But can we reliably distinguish between AO's resulting from such contamination of the generating process and those that simply reflect sampling fluctuations from a rather fat-tailed distribution?

The paper's equations (4.2) and (4.3) define a single AO or IO, though other types of IO are likely to be needed, e.g.,

$$\text{IO(ii)}: \; z_t = \frac{\theta(B)}{\phi(B)} a_t + \frac{\alpha_0}{1-B} \xi_t(t_0) \qquad \text{(Their eq. 4.13)}$$

$$\text{IO(iii)}: \; z_t = \frac{\theta(B)}{\phi(B)} a_t + \alpha_0 \frac{1-B}{1-B^s} \xi_t(t_0)$$

IO(iii) represents an abrupt shift in the seasonal pattern between the months (or quarters) at t_0 and $(t_0 + 1)$, which is repeated in later years. (Note: The dummy ω_0 in HBT has been replaced by α_0 to avoid confusion with ω, the spectral frequency.)

4.2 AO Identification

We now examine the relation between the HBT identification of AO's and the traditional method. Their regression equations (4.6) for a single AO at t_i may be written

$$\pi(B)z_{t_i} = e_{t_i} = \alpha_i + a_{t_i}$$

$$\pi(B)z_{t_i+1} = e_{t_i+1} = \alpha_i\pi_1 + a_{t_i+1}$$

$$\pi(B)z_{t_i+2} = e_{t_i+2} = \alpha_i\pi_2 + a_{t_i+2} \ldots$$

Let $p_i' = (0, \; \ldots \; 0, \; 1, \; \pi_1 \; \ldots \; \pi_{n-t_i})$ (n vector)

where n = number of terms in the series. Equations become

$$\mathbf{e} = \alpha_i \mathbf{p}_i + \mathbf{a}$$

If there are k outliers at $t_1, t_2 \ldots t_k$, define

$$P = (\mathbf{p}_1, \mathbf{p}_2 \ldots \mathbf{p}_k) \qquad (n\times k \text{ matrix})$$

and

$$\boldsymbol{\alpha}' = (\alpha_1, \alpha_2, \ldots \alpha_k)$$

Regression equations become

$$\mathbf{e} = P\boldsymbol{\alpha} + \mathbf{a} \tag{4.1}$$

and the normal equations are $P'P\boldsymbol{\alpha} = P'\mathbf{e}$. If the model spectrum is decomposed so that the irregular component is white noise,

$$g_z(\omega) = q_0 + g_m(\omega) + g_s(\omega)$$

the filter for $\hat{I}_t$ is the time-domain equivalent of $q_0/g_z(\omega)$, that is,

$$q_0 \frac{\phi(B)\phi(F)}{\theta(B)\theta(F)} = q_0\pi(B)\pi(F)$$

But this is calculated in our signal extraction program from the trend and seasonal filters as $\{w_0 + w_1(B+F) + w_2(B^2+F^2) \ldots\}$. So, equating coefficients

$$\left.\begin{aligned} q_0(1 + \pi_1^2 + \pi_2^2 \ldots) &= w_0 \\ q_0(1 \cdot \pi_j + \pi_1 \cdot \pi_{j+1} \ldots) &= w_j \end{aligned}\right\} \tag{4.2}$$

Hence,

$$q_0P'P = \begin{bmatrix} w_0 & w_{t_2-t_1} & w_{t_3-t_1} \cdots \\ w_{t_2-t_1} & w_0 & w_{t_3-t_2} \cdots \\ & \cdots & \end{bmatrix} = W \text{ (say)}$$

Also

$$\begin{aligned} \hat{I}_t &= q_0\pi(B)\pi(F)z_t \\ &= q_0\pi(F)e_t \end{aligned} \tag{4.3}$$

So,

$$q_0(P'\mathbf{e})' = (I_{t_1}, I_{t_2}, \ldots I_{t_k}) = \mathbf{I}_0' \text{ (say)}$$

and the normal equations for the regression become

$$W\hat{\boldsymbol{\alpha}} = \mathbf{I}_0 \tag{4.4}$$

Note that the finite sums in $P'P$ have been replaced by infinite series in (4.2), but this is valid because $e_t = 0(t > n)$.

The covariance matrix of $\hat{\alpha} = (P'P)^{-1}\sigma_a^2 = q_0 W^{-1}\sigma_a^2$

For an isolated outlier

$$\hat{\alpha}_i = I_{t_i}/w_0$$

$$\text{SE}(\hat{\alpha}_i) = (q_0/w_0)^{1/2}\sigma_a$$

and the t-ratio is $I_{t_i}/(q_0 w_0)^{1/2}\sigma_a$.

The initial sum of squares of the dependent variable in (4.4) $= q_0 \sum_1^n e_t^2$ and the initial selection of outliers for the regression is made with σ_e instead of σ_a in the t-ratio. Revised residuals from (4.1) are needed to identify further outliers, but instead we can use revised $\hat{I}_t$ from (4.4)

$$\hat{\mathbf{a}} = \mathbf{e} - P\hat{\alpha}$$

Revised

$$\mathbf{I} = q_0\pi(F)\hat{\mathbf{a}} = \text{old } \mathbf{I} - W_*\hat{\alpha}_*$$

where W_* is $(n \times n)$, i.e., W padded with zeroes, and $\hat{\alpha}_*$ is similarly related to $\hat{\alpha}$. We conclude that, provided (4.3) is satisfied, the intuitive identification of outliers through the $\hat{I}_t$ is correct, and so is the amplifying factor $1/w_0$ for isolated outliers in B/E (1965). But this factor now allows for distortion by the outlier of the seasonal component, as well as the trend, and for all interactions between neighboring outliers. After removal of outliers, (4.3) shows that $\hat{I}_t$ is normally distributed with variance $q_0^2(1 + \pi_1^2 + \pi_2^2 \ldots + \pi_{n-t}^2)\sigma_a^2$, which, for terms not too near the end of the series, approximates to $q_0 w_0 \sigma_a^2$. However, towards the end, the truncation of $\pi(F)$ leads to $\hat{I}_t$ having smaller variance; and, by symmetry, the same must be true near the beginning, i.e., the e_t have variance less than σ_a^2.

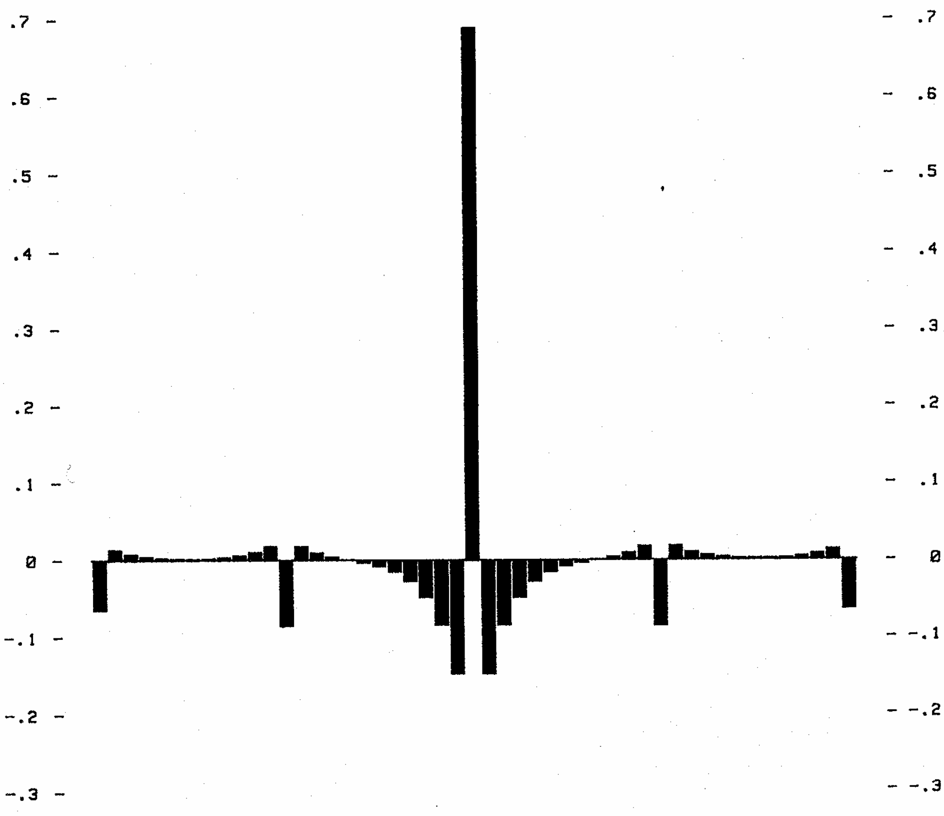
The interaction between outliers is negligible if they are more than 3 months (quarters) apart, except when in the same month of neighboring years. (See figure 1.)

4.3 Estimation of the Outliers

The authors (HBT) recommend an inner iteration to identify outliers and an outer iteration of the whole model, simultaneously reestimating the outliers. We have implemented their inner iteration in slightly modified form so that new outliers enter the regression as they are identified, but the removal of those that have become insignificant is deferred

Figure 1. FILTER WEIGHTS FOR IRREGULAR COMPONENT

$(011)(011)_{12}$ model with $\theta_1 = 0.57$, $\theta_{12} = 0.75$



until the end of the iteration. This change makes no difference when outliers are effectively isolated (W is diagonal), but it avoids the risk of their oscillating in and out of the regression. The simplicity of equation (4.4) permits simultaneous entry of several outliers.

An important philosophical point arises in the use of simultaneous estimation of outliers in HBT's outer iteration. If the outliers are linked with external causes (as they will always be for IO), the hypothesis concerning them is precise and closed, so that simultaneous estimation is justified (as in the case of trading-day adjustment). But, if not, the hypothesis is open ended; that is, one can choose the cutoff point arbitrarily, and in some cases this could lead to an unstable situation with an ever-increasing number of outliers being identified. Moreover, AO's without known external causes may be identified one year and not the next year if they are close to the cut-off point.

An example of the difficulties that can arise with the automatic identification of outliers is provided by the series for United Kingdom unemployed. This is normally very smooth and a (111) $(011)_{12}$ model is fitted very well. The exceptional winter of 1962-63 coincided with the cyclical and, of course, the seasonal peak. The irregular series shows a large positive in March and an even larger negative in April 1963. The program identifies the latter as an outlier, but treats the former as its shadow. What happened was that unemployment rose more than seasonally in January, February, and March and fell suddenly when the snow melted. Knowledge of the external cause would have led to a better treatment of the outliers, as occurred automatically with the less flexible trend of B/E (1965).

A compromise, which I plan to adopt, is—

(i) include all IO and AO with known causes in the model reestimation (once)
(ii) estimate other identified AO from (4.4), and modify the series, but do not reestimate these outliers in the model; also provide for tapered modification, as in X−11 and B/E (1965).

I have serious doubts about the usefulness of HBT's IO in their equation (4.3). If the outlier has produced abrupt changes in both trend and seasonal components, would we be justified in assuming that the structural parameters of the model are unchanged?

4.4 Implications for the spectral decomposition

The argument from the character of the spectra only allows us to conclude that the irregular component should be a low-order MA. The best chance of identifying outliers seems to be by making the spectrum as smooth as possible, reinforcing the minimum signal extraction principle. For this reason, Burman (1980) proposed, for bottom-heavy models, the transfer of a function $\epsilon_1(1 + \cos \omega)$ from trend to irregular. But we now know that valid identification of outliers can be made only with a white noise irregular. This settles the decomposition of the spectrum on the first round. Of course, on the second round, one could have a MA irregular if one is not iterating the identification of outliers. A MA irregular arises naturally in top-heavy models; but, as remarked above, it can lead to an invalid decomposition in which the irregular component of the spectrum takes negative values. I am therefore now persuaded that this component should always be white noise.

5. REVISIONS

The authors remark at the beginning of section 5, "we do not feel that one method . . . should be preferred over another on the basis of revision." I think it is fair to say that if on average one method produces smaller revisions without any sacrifice of quality (e.g., traces of residual seasonality), it should be preferred. They show that for the model-based method θ_{12} is nearly always greater than 0.4, which corresponds to the model for which the X−11 central filters would be approximately optimal. This explains why, on average, signal extraction produces smaller revisions. But the appearance of the seasonal component of the latter is often more flexible than X−11, even in the range $0.4 < \theta_{12} < 0.6$, which must be due to the operation of the noncentral filters.

An important cause of revisions is the fall in $\hat{\theta}_{12}$ resulting from the modification of outliers. This also tends to increase the variation of $\hat{\theta}_{12}$ over different lengths of a series. Columns 1, 2, 5, and 6 of table 1 show how the estimates vary for two series as they are extended from 8 to 19 years. These results were obtained with an earlier version of our signal-extraction program, in which modifications were tapered. It would be useful if the authors examined their extensive data to see whether the use of model estimation of outliers (implying no tapering) has increased the size of revisions.

6. PRIOR PROBABILITIES

For shorter series—under 10 years—the variation of $\hat{\theta}_{12}$ is often much larger than its asymptotic standard error, which must contribute to the revisions. Following Akaike's Bayesian approach, it is suggested that the user be allowed to impose his/her own prior beliefs on the model, namely that the seasonal pattern evolves slowly over time, except when there is a sharp change (IO(iii) above) induced by an external cause. This implies that values of θ_{12} close to 1 should be given higher priors than those in the region 0.4-0.6. We have implemented an example of such a prior function in our program. The prior equals 1 at $\theta_{12} = 1$, is close to 1 for $\theta_{12} > 0.8$, and is set by the user to a value α at $\theta_{12} = 0.6$.

Columns 3, 4, 7, and 8 of table 1 show $\hat{\theta}_{12}$ estimated with priors, setting $\alpha = 0.2$. The effect is to increase $\hat{\theta}_{12}$ but by decreasing amounts as the series lengthens. It also reduces the differences between preliminary and final values and their variation over the span of years. But $\hat{\sigma}_a$ and the Ljung-Box Q are virtually unchanged compared with straight ML, which must mean that the likelihood surface is fairly flat in the neighborhood of the maximum, for series of the length available in economics. So a seasonal prior should cause a signifi-

Table 1. VARIATION IN ESTIMATES OF θ_{12}

	Series 1[1]				Series 2[2]			
	ML		ML + priors (α = 0.2)		ML		ML + priors (α = 0.2)	
Number of years	Prelim.	Final	Prelim.	Final	Prelim.	Final	Prelim.	Final
8	.682	.640	.754	.808	Fixed	Fixed	Fixed	Fixed
9	.699	.648	.751	.767	.951	.867	.966	.890
10	.744	.776	.781	.818	.905	.775	.923	.812
11	.785	.874	.807	.893	Fixed	Fixed	Fixed	Fixed
12	.843	.886	.860	.927	.925	.884	.934	.891
13	.788	.786	.805	.791	.844	.812	.858	.847
14	.780	.701	.795	.736	.826	.778	.838	.808
15	.810	.756	.819	.779	.751	.651	.774	.702
16	.758	.621	.772	.666	.749	.663	.770	.704
17	.772	.647	.784	.680	.801	.748	.811	.766
18	.812	.703	.817	.724	.826	.795	.833	.809
19	.820	.712	.825	.732	.839	.813	.844	.824
Mean	.774	.729	.798	.777	.868	.816	.879	.838
SD	.0458	.0849	.0300	.0748	.0835	.1056	.0790	.0923

[1]United Kingdom production of passenger cars.
[2]United Kingdom engineering orders on hand.

cant reduction in the size of revisions without any loss of quality of seasonal adjustment.

In a similar way, priors may be introduced for the smoothness of the trend; but the corresponding parameter estimates seem to be little affected, and I am not sure yet whether such priors will prove useful. Details of the method will be published elsewhere.

7. CONCLUSIONS

I should like to thank the authors for providing extensive insights into the practical problems of applying signal extraction, e.g., trading-day and Easter adjustments, and in particular for clarifying the statistical treatment of outliers and the related question of the canonical decomposition.

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COMMENTS ON "MODELING CONSIDERATIONS IN THE SEASONAL ADJUSTMENT OF ECONOMIC TIME SERIES" BY S. C. HILLMER, W. R. BELL, AND G. C. TIAO

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PRAISE WHERE PRAISE IS DUE

The paper by Hillmer, Bell, and Tiao (1982) is a very substantial one. It leaves this reader convinced that their approach to seasonal and calendar adjustment—modeling the trend, seasonal, and irregular by ARIMA models, modeling the calendar in the X-11 manner by a linear combination of day-of-the-week counts, and using signal extraction to carry out the adjustments—is one which deserves serious consideration.

A HISTORICAL NOTE

Grether and Nerlove (1970) were the first to suggest modeling trend, seasonal, and irregular components by ARIMA models and using signal extraction to carry out seasonal adjustment. This work culminated in a book (Nerlove, Grether, and Carvalho 1979), several chapters of which are devoted to such modeling. It will be important for us to compare the Hillmer-Bell-Tiao technology with the Nerlove-Grether-Carvalho technology, both theoretically and empirically.

THE DESIDERATA

As the authors have clearly stated, there is an inevitable arbitrariness to the definition of the seasonal component. When seasonally adjusting a series, in isolation, without ascribing the seasonal variation to some specific cause, the best we can do is simply say that we want to remove variation at and near the seasonal frequency (1 cycle per 12 months for monthly data with a yearly seasonal component) and its harmonics; just what "near" means is and must be somewhat arbitrary.

The situation with calendar adjustment is different; here we are addressing the variation associated with the number of times different types of days occur in each month, so that specific explanatory variables can be defined and regression techniques used to remove the variation. Of course, there is the usual vagueness about the specific form of the explanatory variables; in fact, I shall argue shortly that a definition of explanatory variables somewhat different from that of the authors is more appropriate.

Even though the definition of the seasonal component is in part arbitrary, there are nevertheless some broad guidelines for the seasonal component and the trend component, which I shall describe for a monthly series with a yearly seasonal.

> The trend is a portrayal of the low frequency, or long term, variation in the data. Thus, it should appear like a smooth curve drawn through the data.
>
> Each monthly subseries of the seasonal—for example, the January values—should describe the low frequency, or long term, variation in the corresponding monthly subseries of the data minus the trend (and minus the calendar component if one is present).

This is the general thinking that has guided seasonal adjustment in the past. The earliest expression of this thinking that I have been able to find is in Macaulay (1931, app. I).

GRAPHICS

Irma Terpenning, Susan Devlin, and I have argued that graphical displays deserve to be a routine part of seasonal and calendar adjustment. We know of no other way of providing such powerful tools for assessing the adequacy of the adjustment process and for understanding the variation in the series. In Cleveland and Terpenning (1982), there are graphical displays for seasonal adjustment and in Cleveland and Devlin (1980), there are graphical displays for calendar adjustment.

I used these displays to investigate the performance of the Hillmer-Bell-Tiao methodology. Three of these displays will be shown later for the following three monthly series:

- Series 1. Number of unemployed males in the U.S., ages 16-19
- Series 2. Natural logarithms of wholesale sales of hardware in the U.S.
- Series 3. Number of nonagriculturally employed males in the U.S.

Series 2 and 3 are analyzed in the Hillmer-Bell-Tiao paper and series 1 is analyzed in Hillmer and Tiao (1982). Steve

Hillmer kindly provided me with their trend, seasonal, irregular, and calendar components for these series. I also ran the SABL seasonal and calendar adjustment procedures (Cleveland, Devlin, and Terpenning 1981) on these series.

The same three graphical displays that are used to show the authors' results are also used to show the SABL results to provide a comparison of the two methods. Figures with odd numbers show the three types of graphical displays of the Hillmer-Bell-Tiao components for the series. Figures with even numbers show the same displays for the SABL components for these three series. The following sections describe the three types of displays used.

THE DATA AND COMPONENTS PLOT

The display shown in figure 1 is a plot of the data and the components in a vertical array of panels, so that time is a common horizontal scale. Each panel has its own scale, chosen so that the plotted values fill the panel. This is done since, typically, the ranges of the series and the components are very different. The drawback to having different scales, of course, is that the relative amount of variation can be seen only by looking at scale labels. To provide a more visual appreciation of the relative variation, a bar has been drawn to the right of each panel. The lengths of the bars represent the same amount of change in each panel. For example, a change in the value of the trend component equal to the length of the trend bar is the same as a change in the irregular component equal to the length of the irregular bar.

SEASONAL SUBSERIES PLOT

The plot of the seasonal component in the data-and-components plot certainly gives much information, but we cannot assess the behavior of each monthly subseries of the seasonal. This can be done in the seasonal subseries plot shown, for example, in figure 3. First, the January values of the seasonal are plotted for successive years, then the February values, and so forth. For each monthly subseries the midmean of the values (the average of all values between the quartiles) is portrayed by a horizontal line. The values of the subseries are portrayed by vertical lines emanating from the midmean line. The predicted values of the seasonal, which are frequently used to adjust data for the coming year as they come in, are portrayed by dashed lines. The seasonal subseries plot allows an assessment of the overall pattern of the seasonal as portrayed by the horizontal midmean lines and also of the behavior of each monthly subseries. Since all of the values are on the plot we can see whether the change in any subseries is large or small compared with the overall pattern of the seasonal.

THE SEASONAL-IRREGULAR PLOT

Earlier I pointed out that each monthly subseries of the seasonal can be thought of as the result of smoothing the corresponding monthly subseries of the data minus the trend. Thus, the values of the former should be as smooth as possible, subject to the constraint of reproducing the overall long term pattern in the latter. One way to judge the performance of the smoothing is to plot, for each month, the monthly subseries of the seasonal and the monthly subseries of the data minus the trend (and minus the calendar component if it is present), which is equal to the seasonal plus the irregular. This has been done, for example, in figure 5. The seasonal plus irregular is plotted using the symbol "O" at the plotting locations; the seasonal is plotted using a connected plot in which successive plotting locations are connected by straight lines; the values of the seasonal predicted 1 year beyond the end of the data are plotted by the symbol "+". Sometimes a "*" is used as a plotting character instead of "O"; at these times the irregular value is very large in absolute value and the seasonal plus a modified irregular has been plotted. This is done so that outliers in the irregular do not destroy the resolution of the display.

I have included the three graphical displays for the three examples described above to give the reader the opportunity to make his or her own judgments. In the following sections, I will make some observations based on these displays.

THE NEED FOR ROBUSTNESS

The authors are quite accurate in stressing the importance of robust estimation to cope with outliers. The detrimental effect of outliers in economic data on standard statistical procedures was noted at least half a century ago (Kuznets 1933). An example of how an outlier can distort results is provided by the decomposition of the unemployed males series shown in figures 1, 3, and 5; Hillmer and Tiao (1982) used the version of their methodology without the robustness capability. The bottom panel of figure 1 shows two large positive outliers in the irregular; as it happens they both occur in May, one in 1965 and one in 1966. If we now look at the May panel in figure 5, we see that the seasonal component has been substantially distorted for this month; in an effort to accommodate the outliers, the seasonal values for May from 1967 to 1971 have been pulled up above what appears to be reasonable for these years.

It would be interesting to see how the authors' robustness methodology performs on this example. Figure 6 shows the SABL robustness methodology was able to cope with the outliers; the seasonal in the May panel has not been distorted for the years 1967 to 1971.

SMOOTHNESS OF THE SEASONAL

Figure 11 shows, for the authors' decomposition of hardware sales, some peculiar behavior: the seasonal subseries all have a small amount of local roughness. The seasonal subseries for the authors' decomposition of employment and unemployment are smoother (see figs. 5 and 17) even though all three models and their parameter estimates are very similar. For these three series, the models are (apart from the fact that hardware has a calendar component)

$$(1 - B)(1 - B^{12})Z_t = (1 - \theta_1 B)(1 - \theta_{12} B^{12})a_t$$

where the parameter estimates are

	$\hat{\theta}_1$	$\hat{\theta}_{12}$
Unemployed	.31	.82
Hardware	.19	.74
Employed	.27	.82

It would be interesting to work out the correlation structure of the seasonal subseries and see if it depends sensitively on the value of $\hat{\theta}_{12}$.

WHY NOT A DETERMINISTIC SEASONAL PIECE?

One strong impression conveyed in the seasonal subseries plots of all of the examples is that the seasonal components consist of a large overall stable seasonal cycle (as represented by the horizontal midmean lines) around which are moderate or minor fluctuations of the seasonal subseries. The pattern seems to beg for a model having a stable deterministic piece. Thus, I strongly second the motion of Pierce (1978) to do this by including a month-of-the-year effect for $E(Z_t)$.

WHY NOT DIVIDE BY MONTH LENGTH?

When the monthly series is an aggregate (flow) for each month, seasonal variation will be induced in the series simply because of changing month length; for example, all other things being equal, February will be less than March. The obvious and simple expedient is to divide the data by month length (before transforming). This has been done in the SABL decomposition of hardware sales portrayed in figures 8, 10, and 12 but was not done in the authors' decomposition portrayed in figures 7, 9, and 11.

It should be remembered that frequently one wants to study the seasonal component to understand its behavior as a way of understanding the behavior of the series. When aggregate data are not divided by month length, the month-length variation goes into the seasonal component; thus the seasonal represents a confounding of the uninteresting month-length fluctuation and the interesting seasonal behavior. In figure 10, where the original data were divided by month length, we do not have the effect of month length as in figure 9, so we can see that some of the interesting seasonal behavior is low values for January and December and a big drop from June to July.

CALENDAR EXPLANATORY VARIABLES SHOULD TAKE ACCOUNT OF HOLIDAYS

The authors' calendar explanatory variables, X_{it}—seven (linearly dependent) variables which are the numbers of times the 7 days of the week occur in each month—are the same as those suggested by Young (1965) and used in X-11 (Shiskin, Young, and Musgrave 1967). If we divide the data by month length, then the explanatory variables should be divided by month length so that they become the fractions of times, $\bar{X}_{it}$, the days of the week occur in a month.

Clearly, it is important to make appropriate modifications of these explanatory variables to allow for holiday effects, some of which would otherwise go into the irregular and, therefore, not be removed in the adjustment process. Remember that β_i is the effect of the i-th day of the week on the aggregated *daily* series. If a particular day is a holiday, it is unlikely that β_i will still be appropriate.

Susan Devlin and I (Cleveland and Devlin 1982) have taken the following approach. Consider a holiday that always occurs on the i-th day of the week and always in the same month, such as Labor Day in the United States. First, since it always occurs in the same month, its effect can be accounted for by the seasonal component. Second, if the calendar modeling proceeds as if the holiday did not occur and the holiday is treated like any other i-th day of the week, then the same β_i is always added each year (e.g., β_1 for Labor Day) by the calendar component. But in the fitting, this will be compensated for by β_i being subtracted from the seasonal. Thus, holidays that are always in the same month and on the same day of the week can be treated like any other day in forming $\bar{X}_{it}$.

Suppose the holiday occurs in the same month, but changes the day of the week; examples are Christmas and January 1. The effect of such a holiday, since it is always in the same month, can go into the seasonal, provided, of course, the effect does not change according to the day of the week on which it occurs. But now, if the holiday is ignored in the calendar modeling and treated like any other day of the week, then the β_i that is added will vary with the year. The effect of this misrepresentation of the holiday as β_i will be incorporated in the irregular, which is undesirable since it will not be removed in the seasonal and calendar adjustment process. The solution is simply not to count the holiday in X_{it}.

THE CALENDAR COEFFICIENTS FOR HARDWARE SALES

I am slightly uncomfortable with the authors' esimates of β_i for the hardware sales data. The values are—

	$\hat{\beta}_i$
Mon	.001
Tue	.013
Wed	.004
Thu	.011
Fri	.001
Sat	−.015
Sun	−.015

I am not surprised to see the Tuesday, Wednesday, and Thursday coefficients higher than those for Monday and Friday; many economic series, such as numbers of phone calls, show this behavior. But the Wednesday value is peculiarly low. Does this make sense? We need to consult a hardware sales expert. However, I do not need an expert to tell me it is not true that Sunday sales are the same as Saturday sales, as the coefficients would seem to indicate. The following are coefficients resulting from the SABL calendar estimation procedures:

	$\hat{\beta}_i$
Mon	.016
Tue	.104
Wed	.066
Thu	.086
Fri	.010
Sat	−.126
Sun	−.156

Could it be that the authors not adjusting the calendar explanatory variables for holidays has resulted in somewhat distorted calendar coefficient estimates?

ROBUSTNESS

The procedure suggested by the authors for dealing with outliers—the IO and AO models, the methods of estimating ω, and the use of likelihood ratio tests to detect the two kinds of outliers—are essentially the same as the procedures suggested by Fox (1972). Martin and Zeh (1980) have compared the Fox procedures with *M*-estimates and *GM*-estimates (Denby and Martin 1979) in the IO and AO cases.

A modification of the authors' implementation of the Fox method might be helpful. As the authors have stated, the estimate of σ^2 can be seriously distorted by outliers. One easy solution, which has been found to work well for robust location estimation and robust regression (Huber 1964; Andrews 1974), is to replace the usual standard error estimate by 1.5 times the median of the absolute residuals. When the data are Gaussian, its expected value is nearly σ. Once all of the outlier modification has been carried out, σ can be estimated in the usual way.

SUMMARY

My suggestions, which I think can improve this already excellent piece of work, are—

- Use graphics to probe the results of the procedures.
- Include stable, deterministic seasonal variation in the model.
- Divide an aggregated monthly series by month length and adjust the calendar explanatory variables appropriately.
- Modify the calendar explanatory variables to account for certain types of holiday effects.
- Use a robust estimate of the standard deviation in place of the usual estimate to avoid contamination by outliers.

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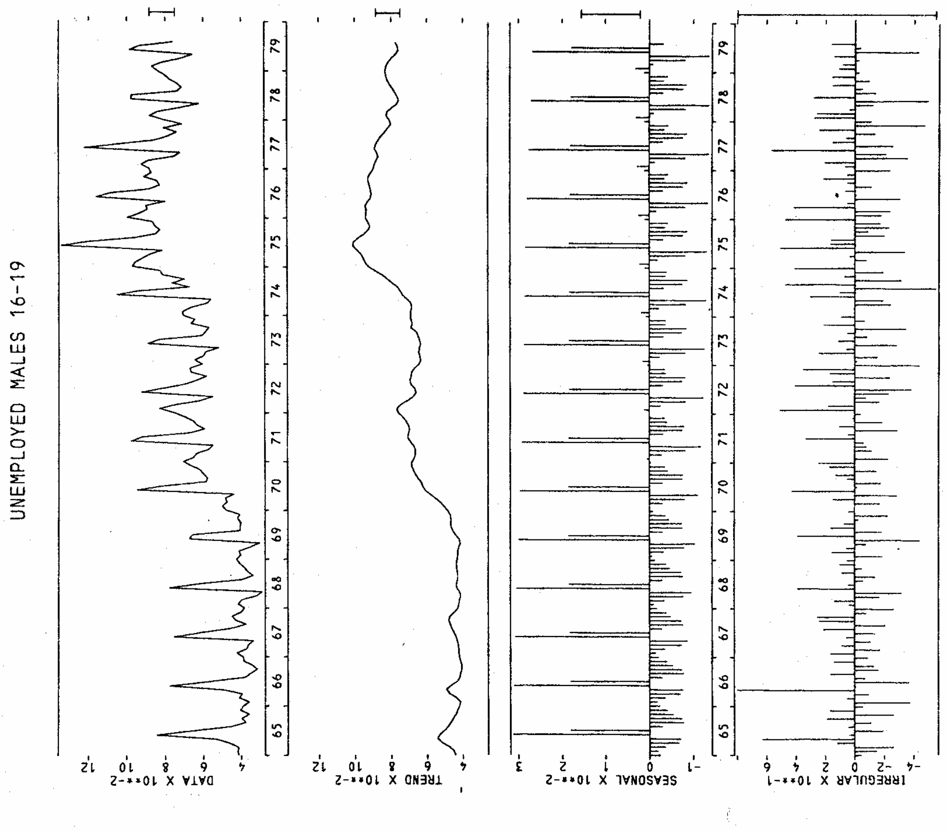


Figure 1. HILLMER-BELL-TIAO DECOMPOSITION

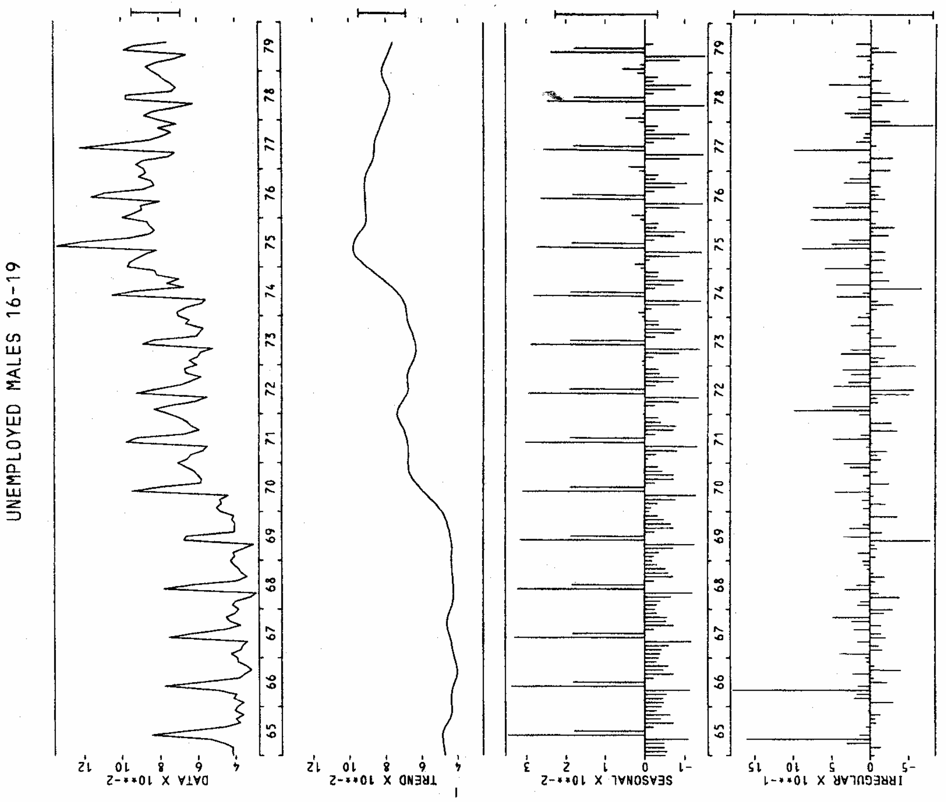


Figure 2. SABL DECOMPOSITION

Figure 3. HILLMER-BELL-TIAO DECOMPOSITION

UNEMPLOYED MALES 16-19

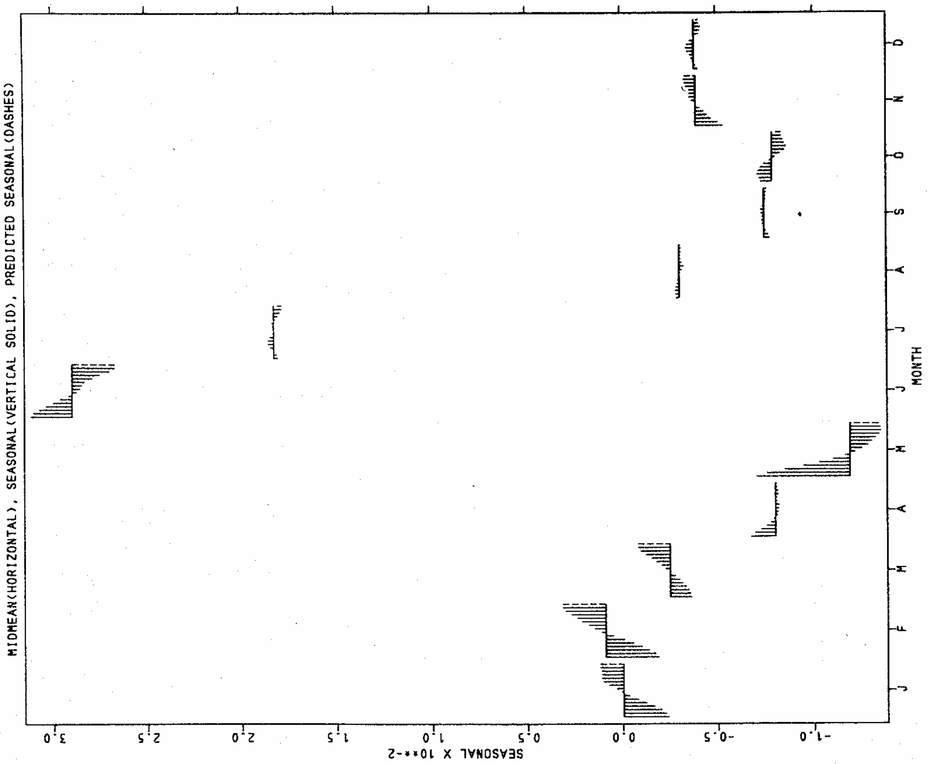
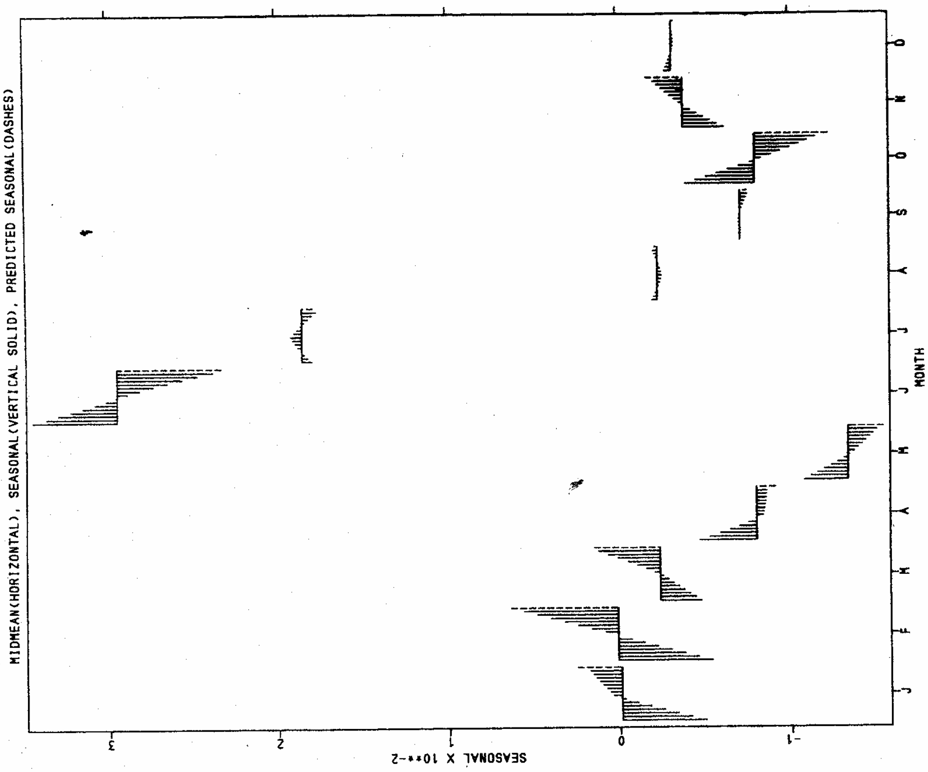


Figure 4. SABL DECOMPOSITION

UNEMPLOYED MALES 16-19



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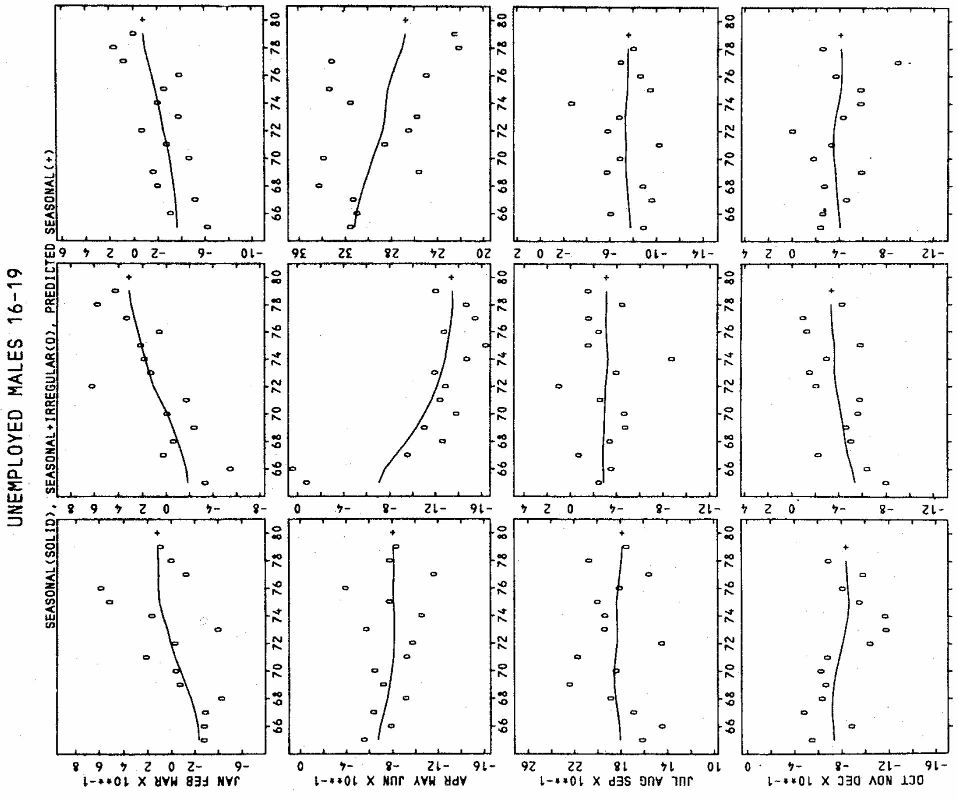

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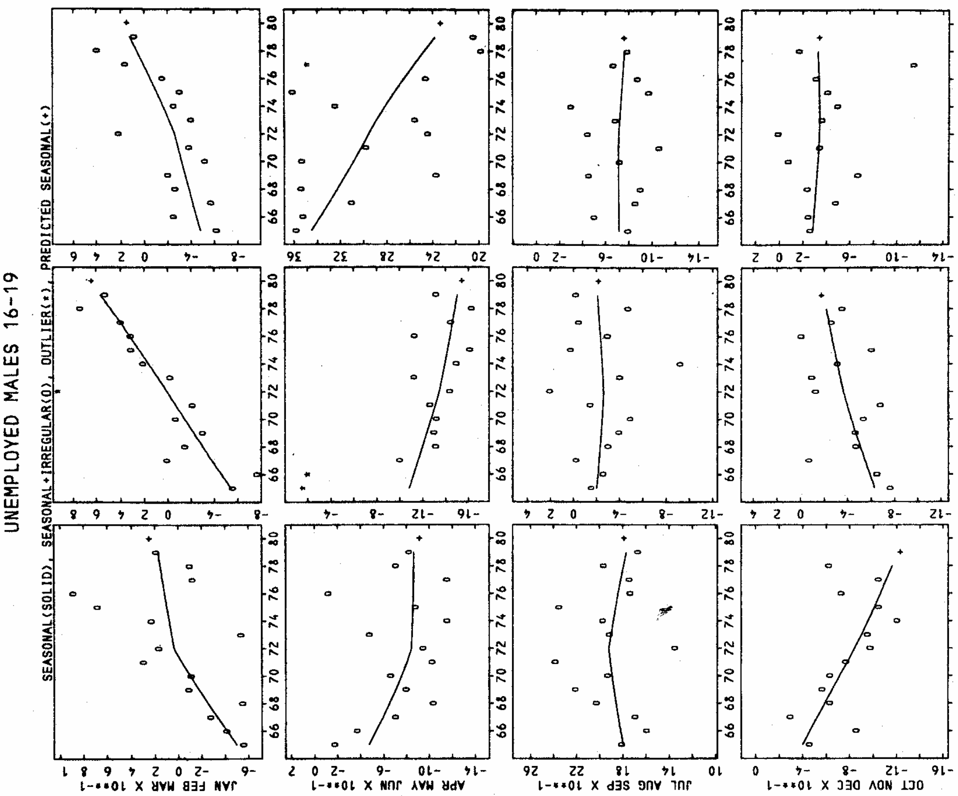

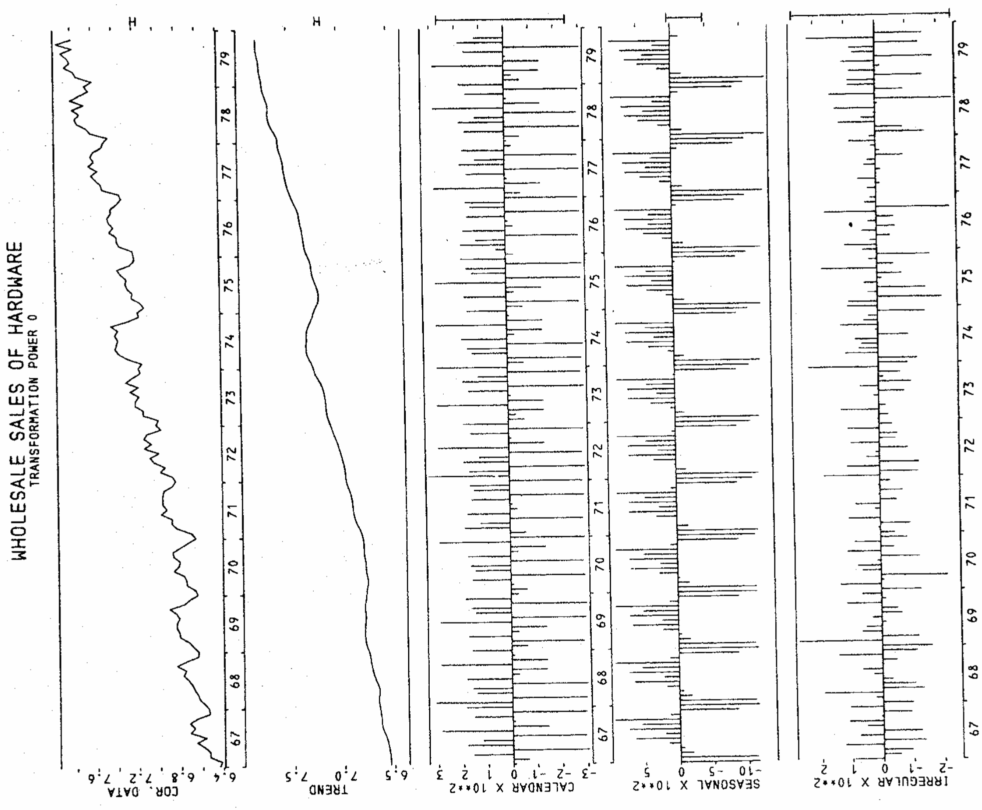

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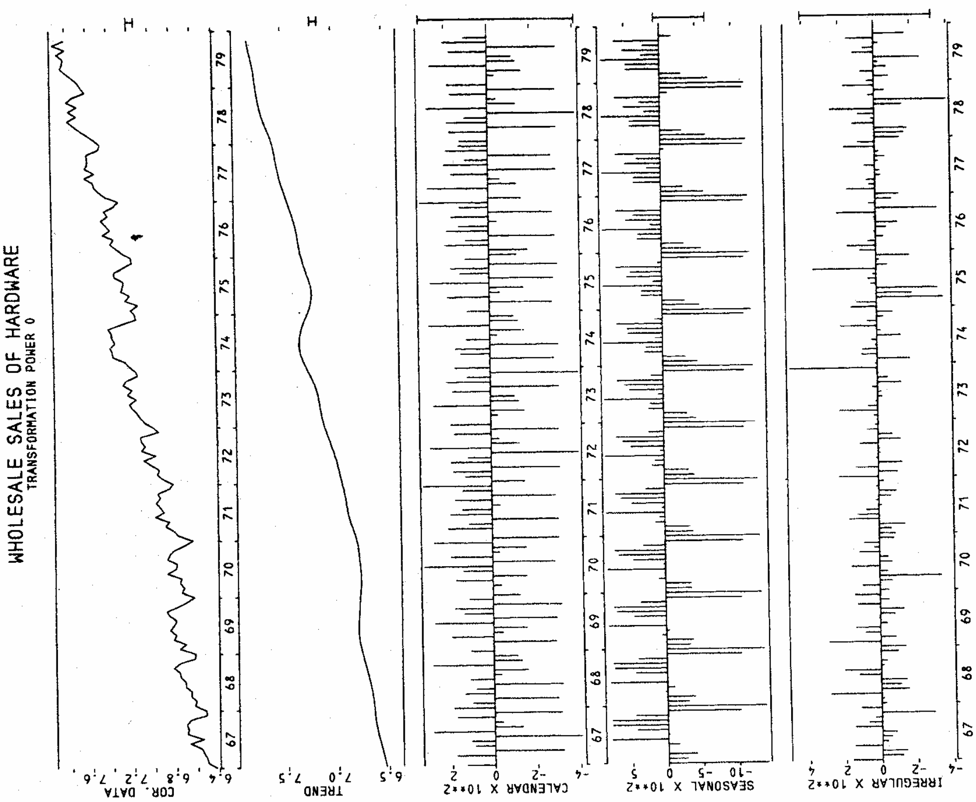
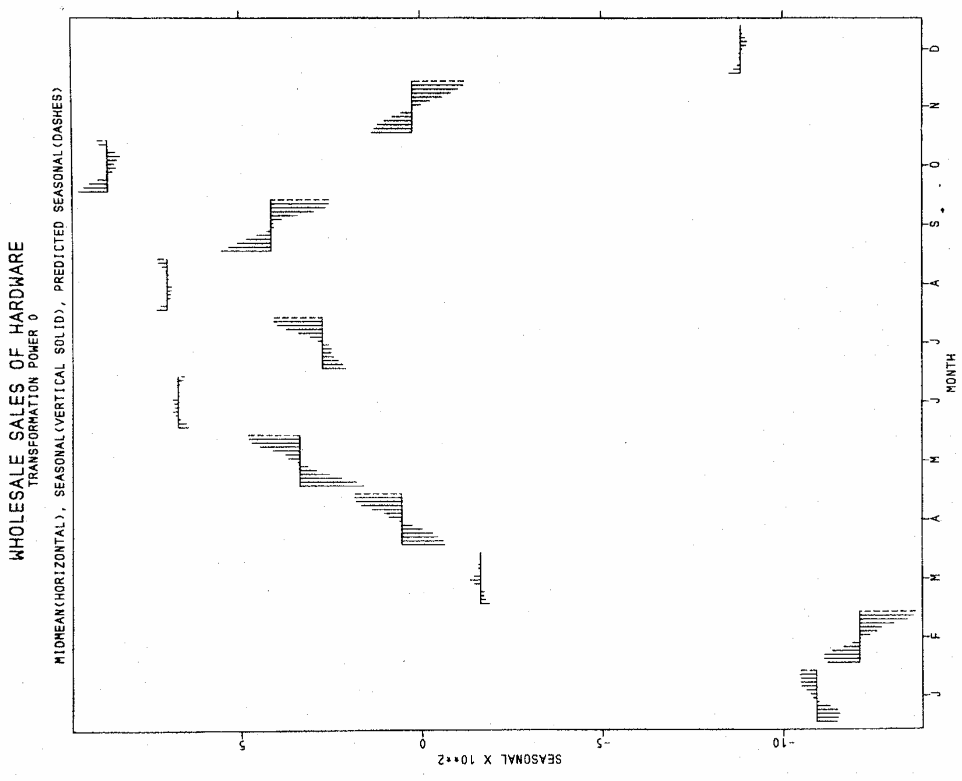


Figure 8. SABL DECOMPOSITION

Figure 9. HILLMER-BELL-TIAO DECOMPOSITION



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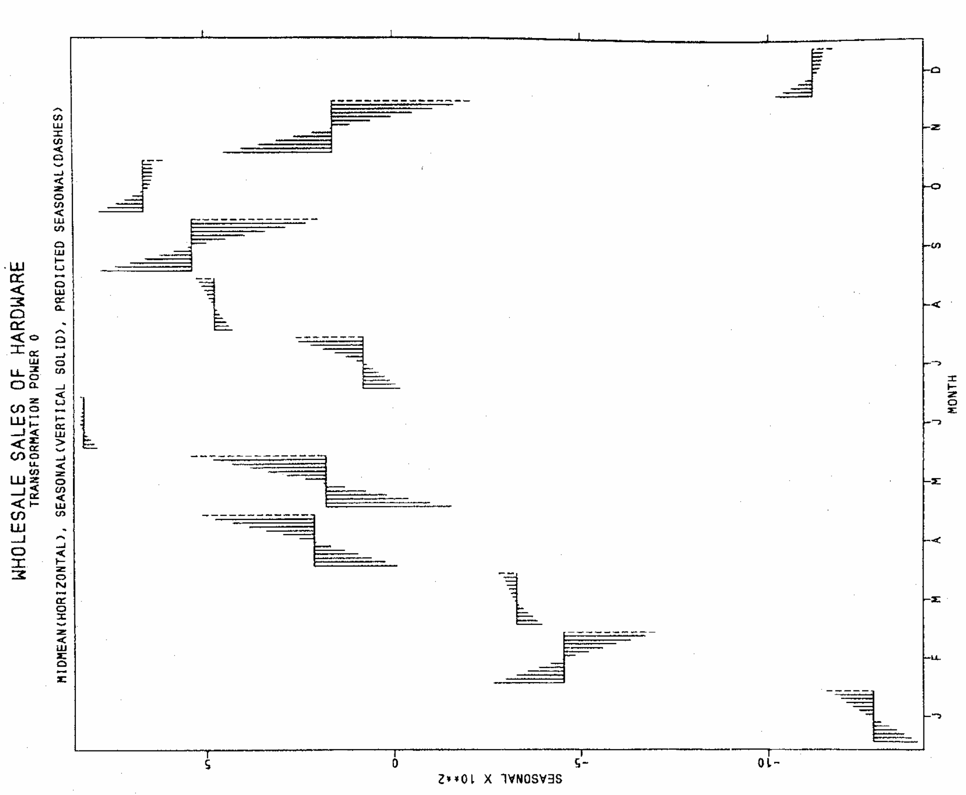

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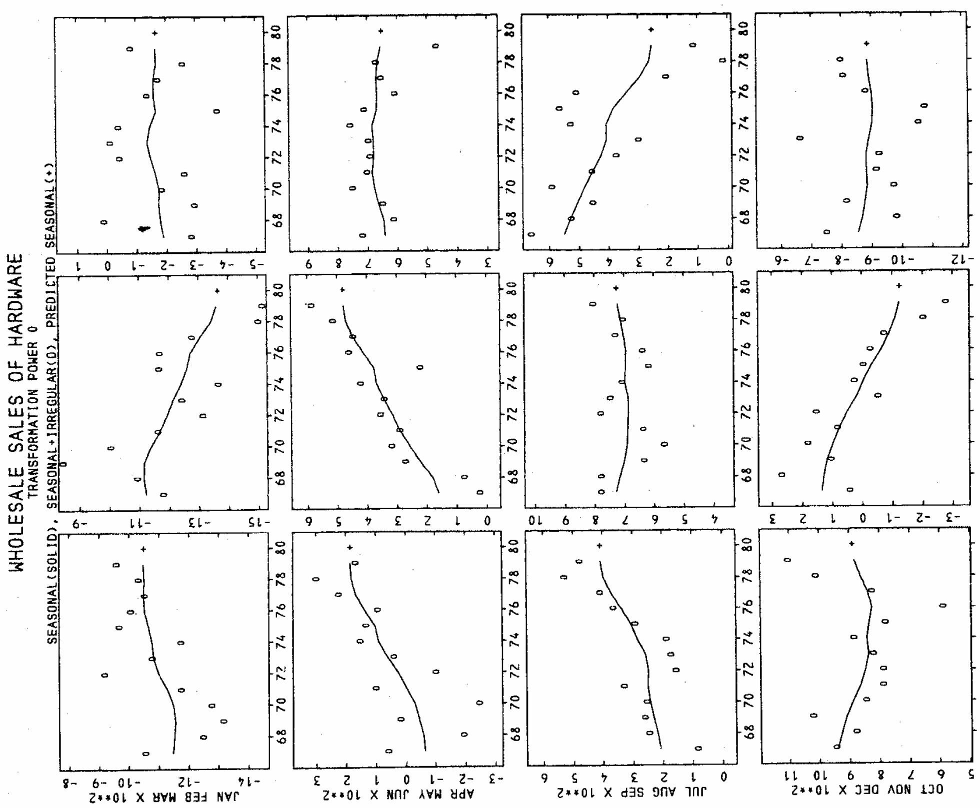


Figure 11. HILLMER-BELL-TIAO DECOMPOSITION

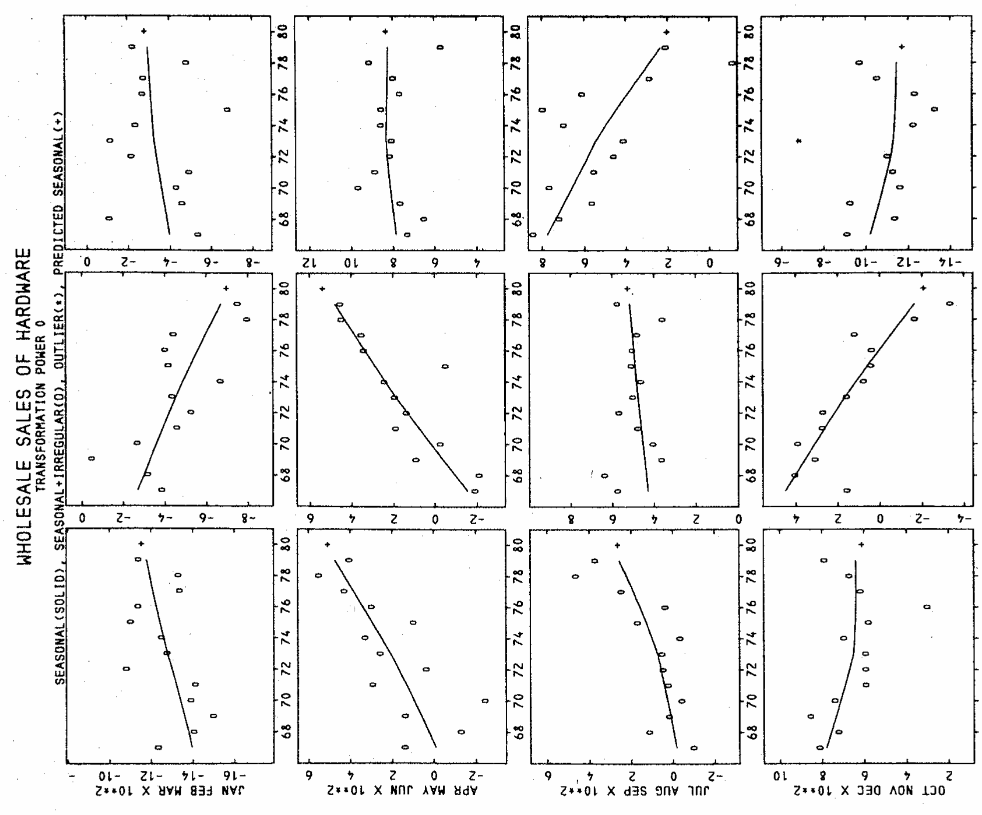


Figure 13. HILLMER-BELL-TIAO DECOMPOSITION

NONAGRICULTURALLY EMPLOYED MALES

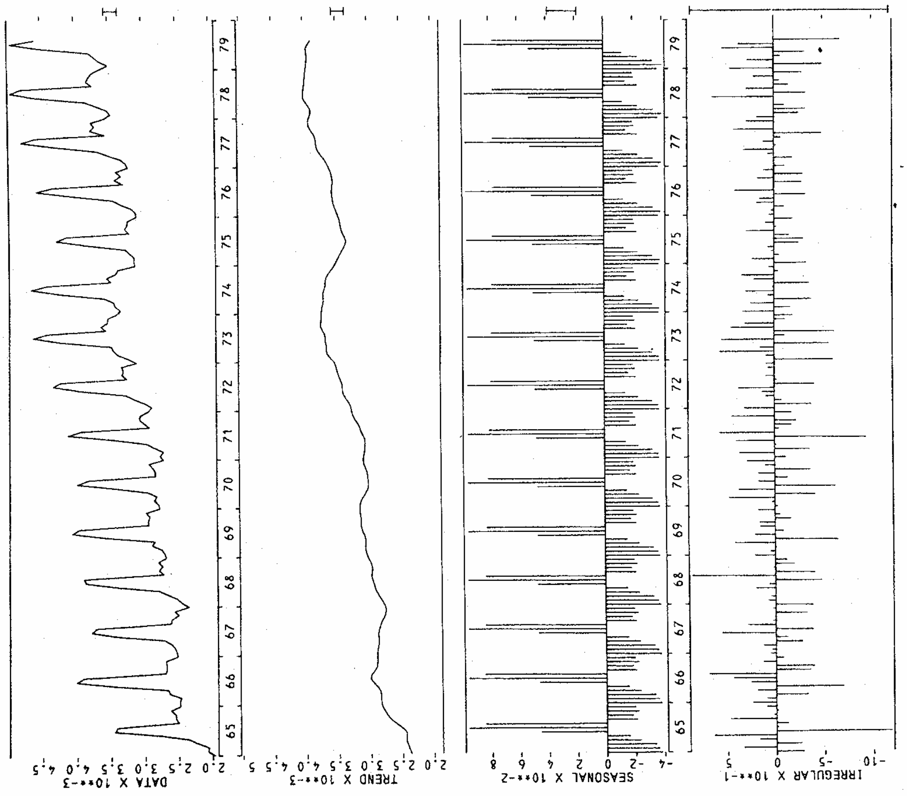


Figure 14. SABL DECOMPOSITION

NONAGRICULTURALLY EMPLOYED MALES

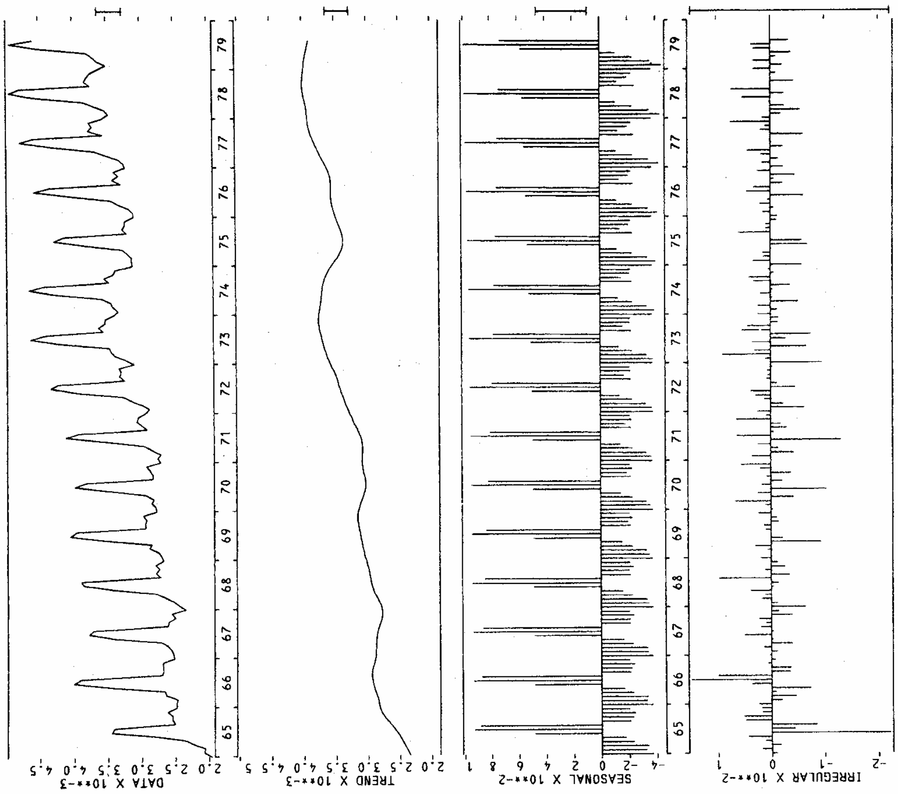
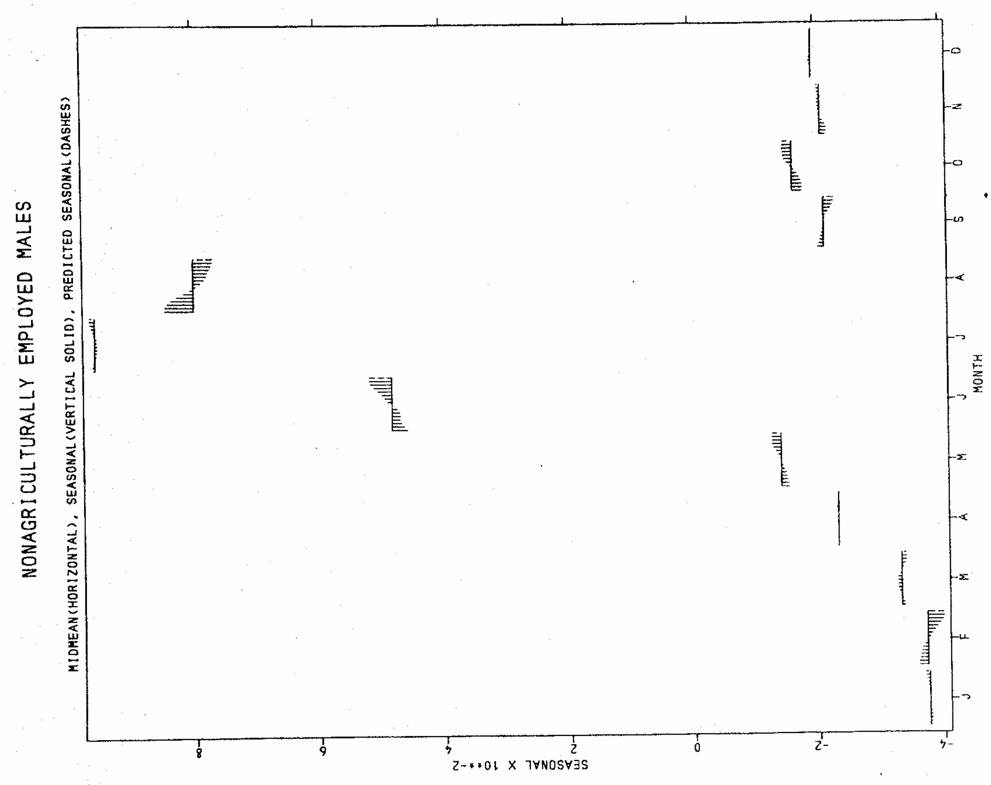


Figure 15. HILLMER-BELL-TIAO DECOMPOSITION



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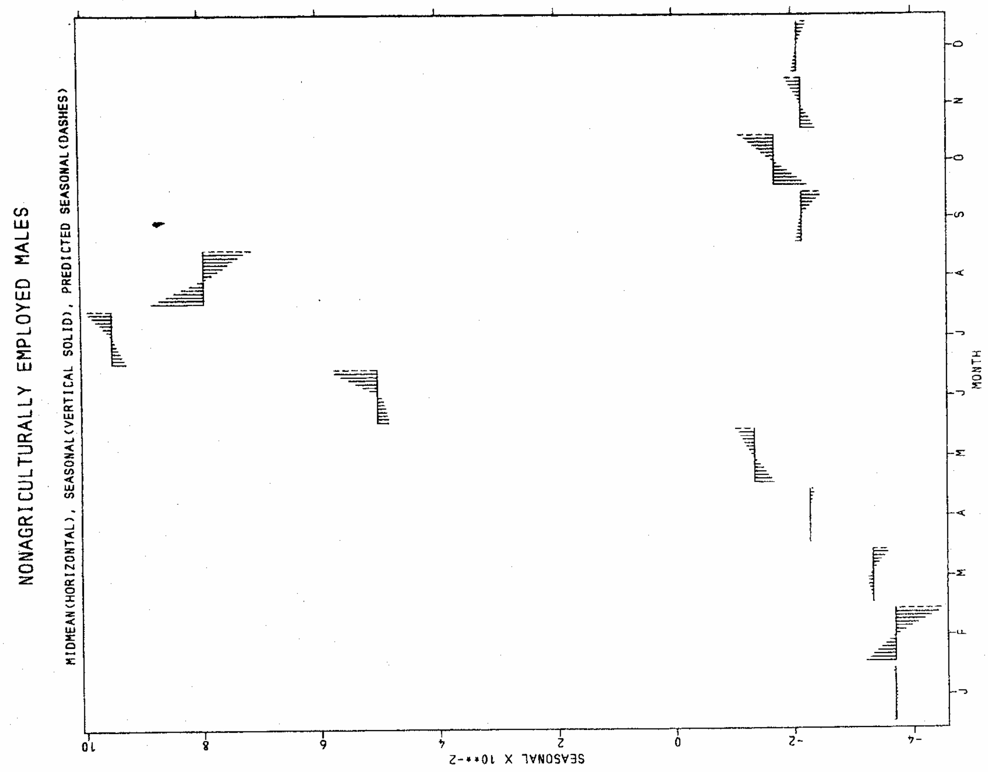

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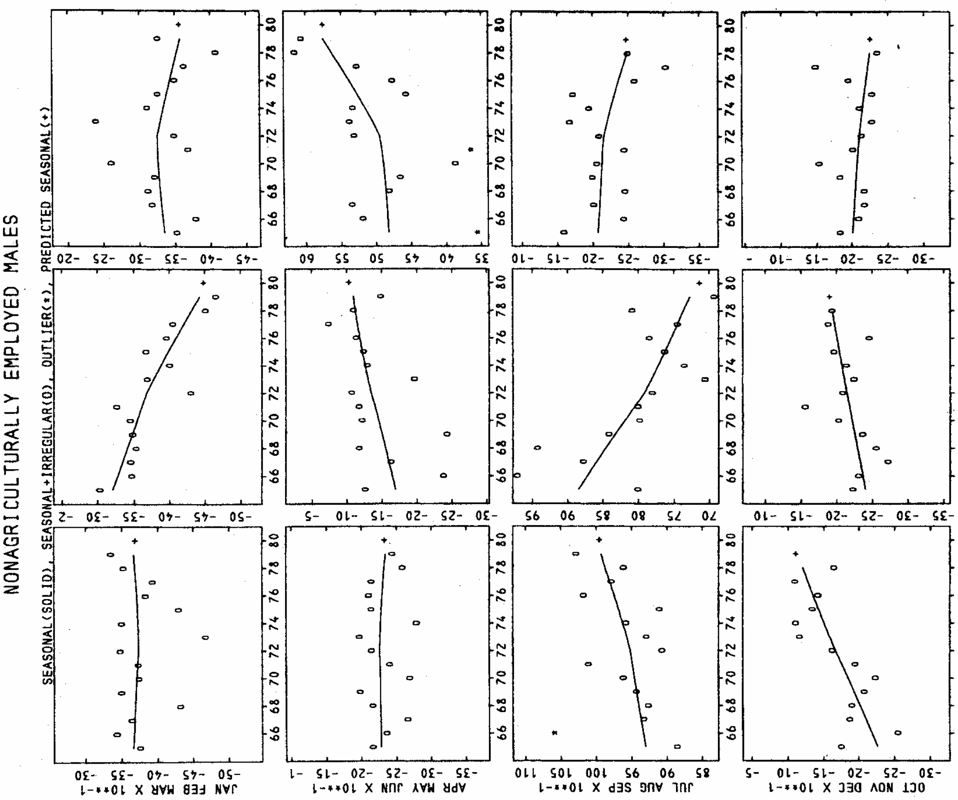
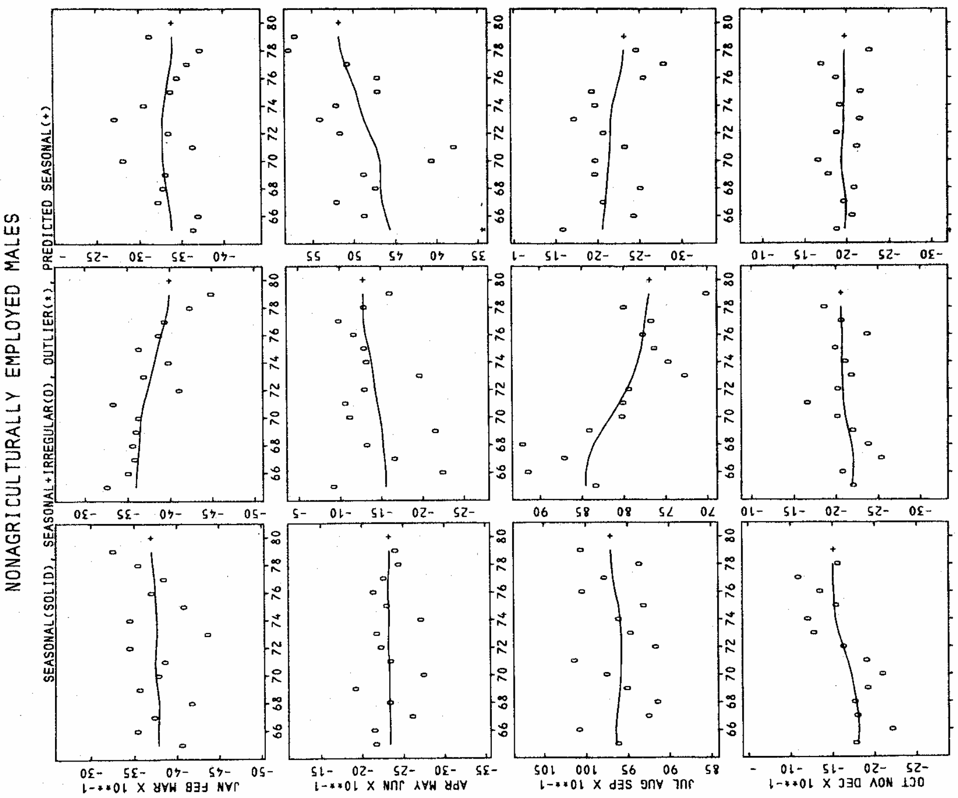


Figure 17. HILLMER-BELL-TIAO DECOMPOSITION



COMMENTS ON "MODELING CONSIDERATIONS IN THE SEASONAL ADJUSTMENT OF ECONOMIC TIME SERIES" BY S. C. HILLMER, W. R. BELL AND G. C. TIAO

Agustin Maravall
Bank of Spain

The authors stress the superiority of a model-based approach to seasonality versus empirical methods. Their approach centers on the use of statistical univariate time series models, yet in adjusting *economic* series there are also *economic* modeling considerations, which can be relevant. They are implied by the implicit demand-supply equilibrium associated with observed quantities and by the existence of economic policy (or control). From the point of view of economic analysis and policymaking, it may be convenient to distinguish between seasonality in demand and supply, as well as between seasonality exogenous to the policymaker and the endogenous one induced by policy.

Although the interest in distinguishing among those different types of seasonality has been occasionally pointed out (Poole-Lieberman 1972, Bach et al. 1976, and Sims 1981), zero attention, however, has been paid to it in practice. (An exception is the work by Plosser 1978.) Very likely, sophisticated and flexible enough estimation methods that permit identification of these seasonal effects are still far from being available to practical adjusters, hence, seasonal adjustment will probably continue to be based on statistical univariate filtering. Yet elementary economic modeling considerations may have relevant practical applications. We illustrate the point through an example derived from a simplified monetary control framework. Seasonal adjustment plays a crucial role in monetary policy where targets are set in seasonally adjusted terms and, therefore, have to be multiplied by their corresponding seasonal factors in order to set the instrument's path. Thus, errors in seasonal adjustment strongly affect the accuracy of monetary control.

Monetary policy is mainly a supply-type control, and exogenous seasonal swings are more likely to come from the demand side. To simplify the discussion, assume there are only two periods in a year. Consider the market of figure 1, where D and S are money demand and supply, respectively, x the rate of growth of the money supply and i the interest rate. (We assume that bank demand for excess reserves is a function of interest rate.) Assume that the monetary authority controls S, shifting it at will in parallel, and that there is an exogenous seasonal shift in D. If D and D' represent money demand in the first and second semester, respectively, the seasonal variation in money will be x_s and the one in interest rate i_s. (The seasonal shift in D could be caused, for example, by a seasonal shift in income.)

Assume a monetary authority whose priority is to avoid interest rate variability (which may depress investment and thereby employment). We shall refer to it as a K-type authority. It shall seek a constant i, for which it shall shift S to S'. Obviously, seasonality in i disappears, while the seasonal move in x increases by Δx_s.

On the contrary, assume the monetary authority to be a very strict monetarist, for which the first priority is to maintain a constant money rate-of-growth. We shall refer to it as an F-type authority. It will shift S to S''; seasonality in x disappears while the one in i increases by Δi_s.

Finally, assume that, at a given time, the monetary authority is changed. The new authority, in order to enforce whatever monetary policy in mind, requests from its staff an estimation of seasonality. The staff, unconcerned with previous monetary policies, simply looks at the univariate information in the x series. (Although simplified, it is nevertheless a fairly accurate description of reality.) Depending upon whether the previous authority was K or F, the conclusion would be: "we are dealing with a very seasonal series" or "no need to worry: there is no seasonality." Obviously the new monetary authority could be in for a surprise.

In general, the correct answer to the monetary authority request needs to consider more information than the one contained in x alone. But there is an interesting case in which the univariate information supplies the correct answer:

Assume a K-type authority. Its interest is to get an answer to the question: By how much does S have to shift (seasonally) so that i remains constant? The answer is: By an amount equal to x_S^T. Further assume that the previous years were characterized by K-type policies. Then the univariate information in the x series would display a seasonal swing equal to x_S^T.

As a matter of fact, real world monetary authorities tend to be of the K-type. For the U.S. case, the objective of accommodating seasonal fluctuations in the demand for credit was specifically stated in the 1913 Federal Reserve Act. For the Spanish case, figure 2 displays the autocorrelation functions of the rate of change of M_3 and of the rate of change of the interbank rate, both series computed as monthly averages of daily values.

However, interest rate series often exhibit some residual seasonality, which seems to have been mostly induced by policy reactions to errors in the estimation of the seasonal com-

Figure 1. SUPPLY AND DEMAND ANALYSIS

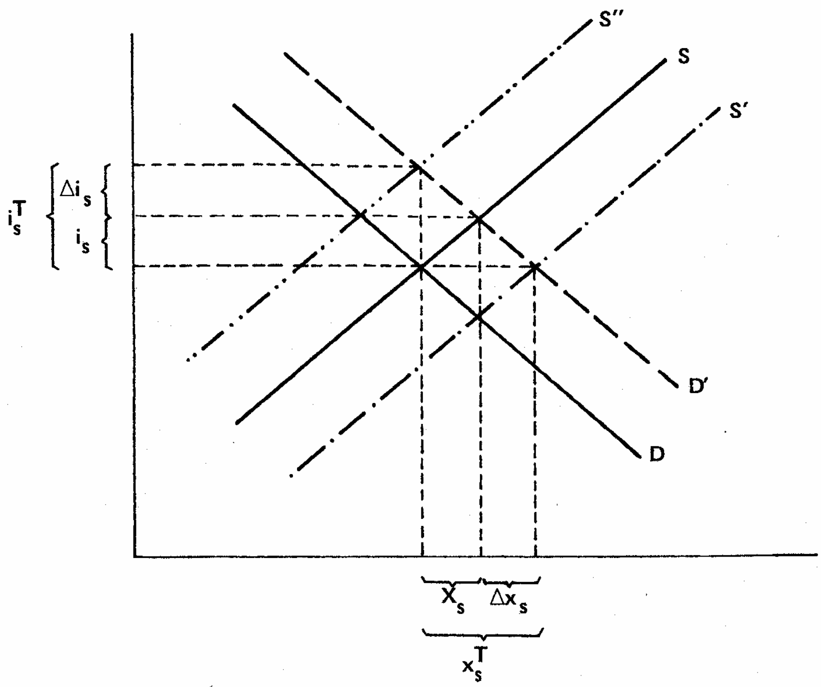
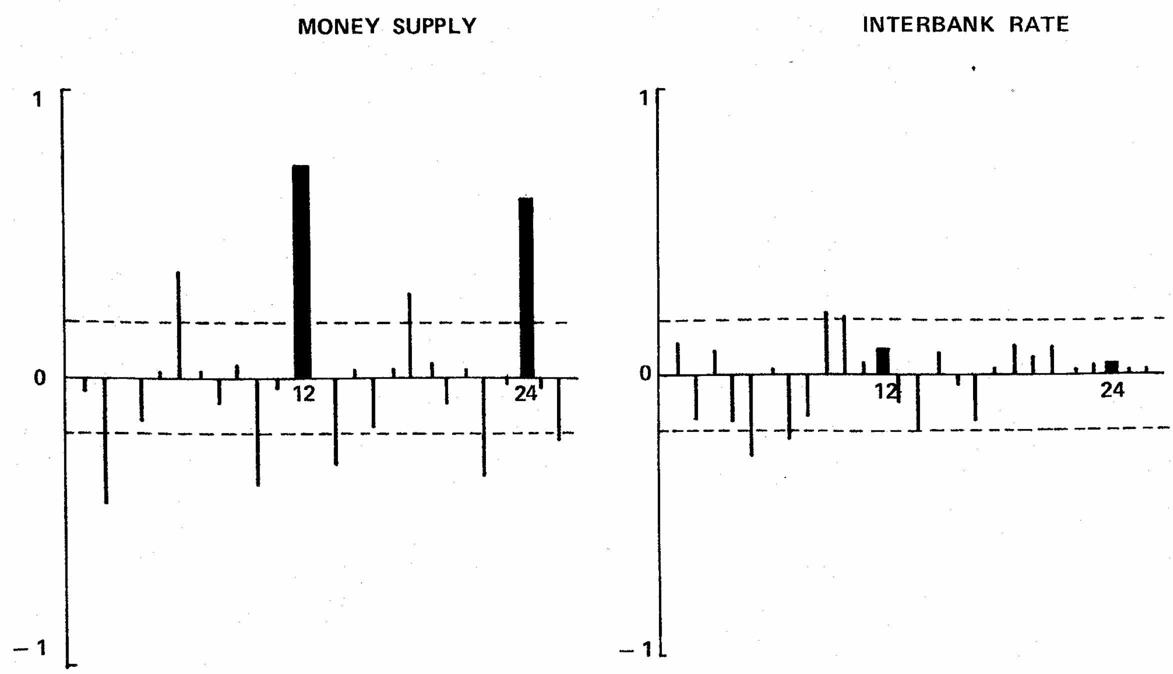


Figure 2. AUTOCORRELATION FUNCTIONS



ponent of the money supply. (See, for example, Lawler 1979 and Maravall 1981.) But, ignoring this small interest rate seasonality, as long as the monetary authority intends to remain K, the use of univariate time series analysis seems valid. Yet, aside from the fact that it simplifies estimation of seasonality, is there a solid reason to prefer a K-type of policy?

Consider, for example, a (partial equilibrium) market, where prices and quantities exhibit seasonal fluctuations. In general, different seasonal patterns imply different sequences of associated welfare measures (such as, for example, the well-known "consumer surplus"). Thus, different seasonal patterns have different welfare properties. Basically, the comparison among them can be done in a manner similar to the analysis of welfare properties associated with price instability. (See, for example, Turnovsky, Shalit, and Schmitz 1980.) It is, then, easily seen that a K-type policy (i.e., removing price variability) may very well be less preferable than a policy which produces a different seasonal profile. In fact, for given D and S and a well-defined optimality criterion, there may be an optimal seasonal path.

Naturally, for such (non-K) policies, seasonal adjustment based on the information of an x series only would be inappropriate. Thus an easy-to-enforce check that could, in some way, indicate to us whether, in order to adjust the money supply, a univariate method is appropriate or not would be quite helpful. We concluded before that univariate seasonal adjustment of the quantity series was appropriate when interest rate series showed no seasonality. This can provide, therefore, the easy-to-enforce pretest. If the i series has no seasonality, univariate statistical techniques can be used. If, on the contrary, there is seasonality in i, a more sophisticated analysis would be required.

Since, as we mentioned before, even under a K-type policy, some seasonality is likely to remain in the interest rate series, the pretest could be roughly equivalent to the one recommended by the authors for determining when a series should be seasonally adjusted (Hillmer, Bell, and Tiao 1981, sec. 2.4). This would amount to the following: If the autoregressive operator of the univariate model for the interest rate series *does not* contain the factor

$$U(B) = 1 + B + \ldots + B^{11}$$

i.e., if no ∇_{12} is needed, then univariate time series methods can be appropriate for adjusting the money supply series, and vice versa. The check is, thus, trivial to compute.

Similar examples can be constructed for different types of markets. In general, for the case of economic variables, when seasonally adjusting a quantity series, information on its dual variable, the price series, should also be considered, and vice versa. (Another important case would be unemployment and wages series.) As we saw before, the dual information might tell us that univariate statistical adjustment is appropriate, or that, in order to get the right answer, a more complete analysis would be required.

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RESPONSE TO DISCUSSANTS

S. C. Hillmer, W. R. Bell, and G. C. Tiao

We would like to thank the discussants for their comments. Burman has shown the relationship between using the model residuals and using the irregulars for (AO) outlier identification. In doing this, he has made us aware of a point which we missed. Consider the case where there is a single AO at time t_i. Burman points out that the vector of computed model residuals $\underline{e} = (e_1, \ldots, e_n)'$ is given by

$$\underline{e} = \omega_A \underline{\rho}_i + \underline{a} \tag{1}$$

where ω_A is the impact of the AO, $\underline{\rho}_i = (0, \ldots, 1, \pi_1, \ldots, \pi_{n-t_i})'$, and $\underline{a} = (a_1, \ldots, a_n)'$ is the vector of innovations. Since the a_t's are white noise, (1) is a standard regression equation and the appropriate estimate of ω_A, its variance, and t-ratio are

$$\hat{\omega}_A = \rho_{t_i}^2 \pi(F) e_{t_i}$$

$$\text{Var}(\hat{\omega}_A) = \rho_{t_i}^2 \sigma_a^2$$

$$\lambda_{2,t_i} = \frac{\hat{\omega}_A}{\text{Var}(\hat{\omega}_A)^{1/2}} = \frac{\rho_{t_i}}{\sigma_a} \pi(F) e_{t_i} \tag{2}$$

where

$$\rho_{t_i}^2 = (1 + \pi_1^2 + \ldots + \pi_{n-t_i}^2)^{-1}$$

In (2) we set $e_t = 0$ for $t > n$. (2) differs from the corresponding result (4.7) in our paper in that $\rho_{t_i}^2$ is used instead of $\rho^2 = (\sum_0^\infty \pi_j^2)^{-1}$. For t_i small enough so that π_j^2 is small for all $j > n - t_i$, $\rho_{t_i}^2$ will be close to ρ^2, so that the results in our paper will be close to those in (2) above. However, for other t_i the λ_{2,t_i} in (2) will be larger than those used in our paper, which will increase the chance of detecting an AO. This difference will obviously be important for t_i near n (how near depending on the π_j's), i.e., at the end of the series. Since detecting outliers in the most recent observations can be important when seasonally adjusting a series, the modification in (2) should be made to our procedure.

Cleveland has suggested that we use 1.5 median ($|e_t|$) in place of the usual estimate of the residual standard deviation in our outlier procedure; this is another useful modification to make. Cleveland has also advocated the approach of Pierce (1978) which involves always removing a deterministic seasonal factor when modeling a series to be seasonally adjusted. We would like to point out that our procedure in fact covers the case of a purely deterministic seasonal component, and that Pierce's definition of a stochastic seasonal component is inconsistent with ours.

For monthly data, a purely deterministic seasonal component means that $S_t = S_{t-12}$ and $U(B)S_t = 0$, which obviously implies that $\text{Var}(U(B)S_t) = 0$. To illustrate that our approach covers this case, consider the model

$$(1 - B)(1 - B^{12})Z_t = (1 - \theta_1 B)(1 - \theta_{12} B^{12}) a_t \tag{3}$$

with $\sigma_a^2 = 1$. Then, from Hillmer and Tiao (1982), an acceptable decomposition is characterized in (4), below, where $\eta_s(B)$ is a polynomial in B and $\eta(B, F)$ is a symmetric polynomial in B and F. In (4) the first term on the right-hand side corresponds to a seasonal component and the second to a nonseasonal component. It follows that the covariance generating function for $U(B)S_t$ is $(1 - \theta_{12})^2 \eta_s(B) \eta_s(F) \sigma_b^2$, and as $\theta_{12} \to 1$ this covariance generating function vanishes implying that Var [$U(B)S_t$] $\to 0$. Noting that for the canonical decomposition Var [$U(B)S_t$] is minimized, we see that as $\theta_{12} \to 1$ the canonical seasonal approaches a purely deterministic seasonal component in which $U(B)S_t = 0$.

$$\frac{(1 - \theta_1 B)(1 - \theta_{12} B^{12})(1 - \theta_1 F)(1 - \theta_{12} F^{12})}{(1 - B)(1 - B^{12})(1 - F)(1 - F^{12})} = \frac{(1 - \theta_{12})^2 \eta_s(B) \eta_s(F) \sigma_b^2}{U(B)U(F)}$$

$$+ \frac{(1 - \theta_1)^2 \theta_{12} (1 - B)(1 - F) + \{\frac{1}{4}(1 - \theta_1)^2 \theta_{12} + \theta_1 \theta_2\}(1 - B)^2 (1 - F)^2 + (1 - \theta_{12})^2 \eta(B, F)}{(1 - B)^2 (1 - F)^2} \tag{4}$$

Cleveland's observation that the seasonal components appear to consist largely of stable seasonal cycles reflects the fact that θ_{12} is fairly large for the examples.

In the situation where the seasonality is changing over time, Pierce (1978) advocates subtracting out monthly means leaving residual seasonality to be modeled as

$$\phi(B^{12})S_t = \theta(B^{12})b_t. \quad (5)$$

Pierce notes that the seasonal autoregressive operator $\phi(B^{12})$ in (5) will typically be a stationary operator. It thus will not contain a factor $U(B)$, so that the way in which Pierce defines stochastic seasonality is not consistent with the definition given in (2.6) of our paper. Therefore, in the presence of changing seasonality these two approaches can lead to different results. The potential difference is a subject for future research.

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