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Frequency Domain Analyses of SEATS and X-11/12-ARIMA Seasonal Adjustment Filters for Short and Moderate-Length Time Series

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We investigate frequency domain properties, revealed by the squared gain and phase delay functions, of short and moderate-length linear seasonal adjustment filters of the ARIMA-model-based signal extraction method of SEATS. X-11/12-ARIMA filters are also considered to a limited extent. The focus is on the one-sided (concurrent) and symmetric (central) filters associated with the Box and Jenkins's airline model for monthly time series of lengths 49 and 109. A Digital Signal Processing perspective on filter properties, favoring interpretability of the filter output, is presented. We show that important features of the finite filters actually used are often not visible in the diagnostics of the infinite filters, such as the Wiener-Kolmogorov symmetric filter gain functions provided by SEATS. For comparing competing adjustments, properties of concurrent filters, especially their phase delays, can be more important than properties of symmetric filters. Our phase delay results illustrate that adjusters who favor smoother seasonal adjustments, or who favor trends for their greater smoothness, must usually reckon with greater delay of turning point and business cycle information for the most recent months. Trend filters are considered briefly. New analytical results are obtained for transfer functions and phase delays.

Key words: Gain function; phase delay function; seasonal adjustment; trend estimation; ARIMA-model-based signal extraction; X-11-ARIMA; X-12-ARIMA.

1. Introduction and Overview

Seasonal adjustment is a signal extraction procedure in which seasonal movements constitute the "noise" that must be suppressed to better reveal the "signal" of interest. In the simplest situation, often achieved by taking logs, the observation Z_t of each month t can be decomposed as $Z_t = S_t + N_t$, where S_t denotes the seasonal component to be suppressed, and N_t denotes the nonseasonal component, i.e., the signal to be estimated.

To obtain such a decomposition for a span of data Z_t, $1 \leq t \leq T$ (from which any detected trading day, moving holiday and outlier effects have been removed), the most widely used seasonal adjustment methods estimate N_t, $1 \leq t \leq T$ by applying linear

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filters. That is, the estimates $\hat{N}_t$ have formulas of the form

$$\hat{N}_t = \sum_{j=-(T-t)}^{t-1} c_{t,j}^T Z_{t-j}, 1 \le t \le T \tag{1.1}$$

with T- and t-dependent filter coefficients $c_{j,t}^T$ that are determined by the methods and software options chosen. In principle, this makes possible the use of standard linear filter diagnostics to analyze properties of these methods, at least for the times t of greatest interest, usually including $t = T$ for the concurrent adjustment. However, the versions of these diagnostics provided by software or presented in the published literature are those for large T, in fact, infinite T in the case of minimum-mean-square-error signal extraction filters from ARIMA models for S_t and N_t, hereafter referred to as ARIMA-model-based (AMB) filters. Also, the diagnostics usually presented are for the symmetric filter that produces the adjustment of the midpoint of a series, which is rarely the time point of greatest interest.

This article studies frequency domain filter diagnostics for monthly series of small and moderate lengths, mainly $T = 49$ and $T = 109$, for filters that produce concurrent and central seasonal adjustments. The main focus is on filters for the canonical ARIMA decomposition of Hillmer and Tiao (1982) utilized by SEATS (Gómez and Maravall 1994) in the situation in which the series to be adjusted is modeled with the airline model of Box and Jenkins (1976) with various typical pairs of coefficient values. This small subset of SEATS's possible filters is adequate for demonstrating the general phenomena and conclusions we wish to present. Also the airline model deserves detailed attention because it is a very frequently used default model of TRAMO and SEATS. (Fischer and Planas 2000 find it adequate for half of 13,222 Eurostat series.) Certain default seasonal adjustment filters of X-12-ARIMA (Findley, Monsell, Bell, Otto and Chen 1998) are also considered. The same filters are available in X-11-ARIMA (Dagum 1980), so we use the hybrid name X-11/12-ARIMA filters. The diagnostics investigated are the squared gain functions of both the concurrent and symmetric filters and the phase delay functions of concurrent filters. Squared gains indicate which frequency components of the data are suppressed or amplified by the filter, and phase delays indicate how frequency components are shifted in time by the filter; see Section 2 as well as Subsection 5 of Appendix A.

As mentioned, filters for the midpoint of an observation interval of odd length $T = 2\tau + 1$,

$$\hat{N}_{\tau+1} = \sum_{j=-\tau}^{\tau} c_{\tau+1,j}^T Z_{\tau+1-j} \tag{1.2}$$

are symmetric, i.e.,

$$c_{\tau+1,j}^T = c_{\tau+1,-j}^T, \quad j = 1, 2, \ldots, \tau \tag{1.3}$$

This is true both for X-11/12-ARIMA with forecast and backcast extensions of equal length and for ARIMA-model-based seasonal adjustments like those of SEATS. In contrast to (1.2)–(1.3), the concurrent adjustment $\hat{N}_T$ is obtained from a one-sided filter,

$$\hat{N}_T = \sum_{j=0}^{T-1} c_{T,j}^T Z_{T-j} \tag{1.4}$$

For AMB signal extraction methods, the nonstationary generalization of Bell (1984) of the Wiener-Kolmogorov formula for the gain function of the symmetric signal extraction filter for bi-infinite data, Z_t, $-\infty < t < \infty$, given as (A.3) in Appendix A, is the only simple[3] general formula available for the gain function of a seasonal adjustment filter directly in terms of coefficients of the models for N_t and Z_t. SEATS provides graphs of this and analogous gains for other components, and in several articles, e.g., Maravall (1999) and Gómez and Maravall (2000), frequency domain properties of these bi-infinite model-based symmetric filters are compared to those of a default symmetric X-11 filter for series of length $T \geq 169$ (without forecast or backcast extension). However, in the published seasonal adjustment literature, there are no frequency domain analyses of SEATS symmetric filters for finite T or of the concurrent seasonal adjustment filters of SEATS for infinite or finite T. Also, this literature does not provide a systematic approach to the interpretation and application of frequency domain filter diagnostics. The present article is intended to fill both of these gaps.

In Section 2, we review definitions and some basic properties of gain and phase delay functions and summarize the literature on gain and phase diagnostics for X-12-ARIMA filters. The core phenomenological results of the article are the diagnostic graphs and detailed discussions of their features presented in Section 3 for a range of SEATS seasonal adjustment filters of various lengths, mainly in Subsection 3.2. Before this, in Subsection 3.1, graphs are shown in which a SEATS filter's diagnostic is plotted together with the diagnostic of a comparable X-11/12-ARIMA filter from the set of filters available for automatic choice by X-11/12-ARIMA. To provide a perspective on the differences between gain and phase delay plots of finite and infinite SEATS filters, the plots for two models' infinite symmetric and concurrent filters are presented in Figure 6 of Subsection 3.2. alongside the plots for the corresponding filters for length $T = 109$. These plots show that it can be impossible to infer important properties of the finite filters from properties of the infinite filters, an important conclusion of this article.

To provide a perspective for evaluating the plots, we present, in A.2 of Appendix A, a Digital Signal Processing (DSP) perspective stressing interpretability of the filter output. For the squared gain functions, our development is based on Wecker (1979). Although there is some overlap, the DSP perspective is different from the perspective of mean square optimal estimation of the unobserved nonseasonal component of the time series. SEATS's method is motivated by the second perspective, but SEATS cannot achieve such optimality, if this is desired, without an unambiguously optimal model. Model ambiguity is illustrated in Section 4 with two models that provide comparable and acceptable fits to a U.S. Census Bureau time series but whose SEATS seasonal adjustment filters have squared gain and phase delay diagnostics which differ in ways that will cause each model

[3] Somewhat more complex and less direct formulas are available for the asymmetric seasonal adjustment filters for semi-infinite data Z_t, $-\infty < t \leq T$; see Bell and Martin (2004).

to be the preferred model for some users. An example of a suboptimal model preferred by experts because of the interpretability of its seasonal adjustment is analyzed with a new squared gain comparison diagnostic for models with different qualities of fit in A.3 of Appendix A. A.5 provides basic results for the interpretation of phase delays and gains.

In Section 5, we briefly consider trend estimation in order to show that concurrent trend filters can have much greater phase delays than concurrent seasonal adjustment filters of the same model, a further illustration of the principle that greater smoothing usually results in greater phase delay, another important conclusion of the article. Section 6 is a summary of our conclusions.

Appendix B describes methods for obtaining the needed coefficients of finite filters. Appendix C establishes that all finite AMB filters for N_t contain the differencing operator of the ARIMA model for S_t. Appendix D applies this result to develop a continuous phase delay function for concurrent filters that reveals fundamental properties of such filters. The Appendices A, C, and D contain the conceptual and theoretical contributions of the article.

2. Gain, Phase and Phase Delay Functions

Using powers of the backshift operator $BZ_t = Z_{t-1}$, a linear filter applied to data Z_t resulting in output $Y_t = \sum_j c_j Z_{t-j}$ can be expressed as the function $C(B) = \sum_j c_j B^j$, i.e., $Y_t = C(B)Z_t$. For monthly data, the *transfer function* of the filter is the generally complex-valued function defined by $C(e^{-i(2\pi/12)\lambda}) = \sum_j c_j e^{-i(2\pi/12)j\lambda}$, $-6 \leq \lambda \leq 6$, when λ is in units of cycles per year. Its amplitude function, $G(\lambda) = |C(e^{-i(2\pi/12)\lambda})|$, is called the *gain function* of the filter. The squared gain $G(\lambda)^2$ has the important property that if Z_t is a stationary time series with spectral density $f_Z(\lambda)$, then the spectral density $f_Y(\lambda)$ of the filter output series Y_t is given by

$$f_Y(\lambda) = G(\lambda)^2 f_Z(\lambda) \tag{2.1}$$

(see Bloomfield 2000, pp. 171–172 for example). Thus, there is suppression of the variance component of frequency λ by the amount of the factor $G(\lambda)^2$ when $G(\lambda) < 1$ and amplification of the component by this factor when $G(\lambda) > 1$. For nonstationary series such as ARIMA processes, there is a generalization of (2.1), (see Subsection A.1), but also ambiguity about how to define frequency variance components (see Wildi 2004). We shall assume that suppression of these components occurs over intervals where $G(\lambda) < 1$, amplification where $G(\lambda) > 1$, and that the effect is larger the farther $G(\lambda)^2$ is from one.

A real-valued function $\phi(\lambda)$ that is defined whenever $C(e^{-i(2\pi/12)\lambda}) \neq 0$ and has the property that $C(e^{-i(2\pi/12)\lambda}) = \pm G(\lambda)e^{i(2\pi/12)\phi(\lambda)}$ holds is a *phase function* of the filter. Here $\pm G(\lambda)$ denotes a real-valued function some of whose values can be negative but whose absolute value is $G(\lambda)$ and which is an even function, $\pm G(-\lambda) = \pm G(\lambda)$ for all λ. Usually, one defines $\phi(\lambda) = (12/2\pi)\arctan(\operatorname{Im} C(e^{-i(2\pi/12)\lambda})/\operatorname{Re} C(e^{-i(2\pi/12)\lambda}))$ to obtain $C(e^{-i(2\pi/12)\lambda}) = G(\lambda)e^{i(2\pi/12)\phi(\lambda)}$, but a different definition with $\pm G(\lambda)$ different from $G(\lambda)$ is sometimes needed to obtain a continuous $\phi(\lambda)$ (see (D.4) of Appendix D). For seasonal adjustment filters, $C(1) = \sum_j c_j = 1$, so $\phi(0) = 0$, and $C(e^{-i(2\pi/12)\lambda}) = 0$ at the seasonal frequencies $\lambda = \pm 1, \ldots, \pm 6$ (see Appendix C). We can restrict attention to the frequencies $0 \leq \lambda \leq 6$ because $\pm G(\lambda)$ is even and phase functions $\phi(\lambda)$ are odd, $\phi(-\lambda) = -\phi(\lambda)$. Gain and phase functions are discussed in many books, e.g., Priestley (1981).

The simplest phase functions are those of the backshift operator and its powers, $B^k Z_t = Z_{t-k}$, for $k = 0, \pm 1, \pm 2. \ldots$. Since the transfer function of B^k is $e^{-i(2\pi/12)k\lambda}$, one can take its phase function to be the continuous function $\phi(\lambda) = -k\lambda$. Note that if $\lambda \neq 0$, then $-\phi(\lambda)/\lambda = k$, which is the delay (when $k > 0$) or advance (when $k < 0$) induced by B^k. In general, when $\phi(\lambda)$ is a continuous phase function for a filter with transfer function $C(e^{-i(2\pi/12)\lambda})$ such that $C(1) > 0$ (so $\phi(0) = 0$), then for every $\lambda \neq 0$ for which $C(e^{-i(2\pi/12)\lambda}) \neq 0$, the value of the function $\tau(\lambda) = -\phi(\lambda)/\lambda$ is interpreted as the (time) *delay*, or the *advance* if $\tau(\lambda)$ is negative, induced by the filter on the frequency component of Z_t with frequency λ. (Subsection A.5 justifies this interpretation of $\tau(\lambda)$ for stationary Z_t.) We call $\tau(\lambda)$ the *phase delay* function. (Rabiner and Gold 1975, p. 80 use this term for $-\tau(\lambda)$, which Wildi 2004, p. 50 calls the time shift.) For example, the seasonal sum filter for monthly data $\sum_{j=0}^{11} B^j$, which is the shortest filter with transfer function zeros at all seasonal frequencies, has the continuous phase function $\phi(\lambda) = -5.5\lambda$ and $\tau(\lambda) = 5.5$, (see (D.4) of Appendix D, where our method for obtaining a continuous phase function for a concurrent filter is described). Phase delays satisfy $\tau(-\lambda) = \tau(\lambda)$. Delay properties are the most useful information conveyed by the phase function, and the phase delay function reveals delay properties more directly, so we use it instead. Except in Appendix D, we only graph phase delays over $0 < \lambda < 1$, i.e., the frequencies of components with periods greater than a year. In fact, the delays of greatest interest are those with frequencies in the interval $0 < \lambda < 1/2$ associated with turning points or with business cycle movements whose dominant components have periods greater than two years. (A trend whose direction reverses sharply at a turning point could have components at higher frequencies.)

Our focus is on SEATS filters because frequency domain properties of X-11-ARIMA seasonal adjustment filters have already been considered extensively. Dagum (1983) provided plots of squared gain and phase functions for default concurrent filters of X-11-ARIMA for series of length $T = 84$ associated with various choices of time series models used to provide twelve forecasts of the series. Her models include several Box-Jenkins airline models. Dagum, Chhab and Chiu (1996) presented spectral analyses of maximum-length concurrent seasonal adjustment, trend and irregular filters from all combinations of 9-, 13-, and 23-term Henderson trend filters and 3×3, 3×5 and 3×9 seasonal filters with forecast extrapolation from airline models as well as without forecast extrapolation. For the case of no forecast extrapolation, Bell and Monsell (1992) showed squared gain function graphs for all of the full-length symmetric X-11 filters. Huot, Chiu and Higginson (1986) analyzed revisions from X-11-ARIMA, defined as the mean square difference between the transfer functions of the central and concurrent filters over all frequencies. For short filters, there is only the unpublished report by Cholette (1979), which provided plots of the gain functions of X-11 concurrent and symmetric filters for the seasonal factors of lengths 36, 48, 60, and 84, together with comments about their phases.

3. Squared Gain and Phase Delay Functions from Airline Models

As mentioned before, the filters we consider are those that arise when an airline model is used to determine the canonical model-based seasonal adjustment filters of SEATS or to extend the data for X-11/12-ARIMA. Box-Jenkins airline models have the form

$$(1 - B)(1 - B^p)Z_t = (1 - \theta B)(1 - \Theta B^p)a_t \tag{3.1}$$

where a_t denotes a series of independent random variables with mean zero and constant variance σ_a^2. For monthly data, $p = 12$. The coefficients θ and Θ, often called the *nonseasonal* and *seasonal* moving average parameters, respectively, are usually constrained to have magnitudes not exceeding one. This entails no loss of generality when the a_t are Gaussian. For seasonal economic time series, the coefficient estimates are usually nonnegative. Thus we only consider (θ, Θ) satisfying $0 \leq \theta \leq 1$ and $0 \leq \Theta \leq 1$.

As an aid to understanding the SEATS filter differences associated with different parameter values, we start by considering how the properties of the time series from (3.1) differ for some extreme coefficient pairs. Given initial values $Z_1, \ldots, Z_{p+1}$, the solutions Z_t, $t \geq p + 2$ of the difference equation (3.1) can be written in the form

$$Z_t = b + ct + s_t + v_t \tag{3.2}$$

with s_t satisfying the "stable seasonality" condition $s_t + s_{t-1} \cdots + s_{t-p+1} = 0$, and therefore $s_{t-p} = s_t$, for $t \geq p + 1$. The quantities b, c, and s_t are functions of the initial values $Z_1, \ldots, Z_{p+1}$, $a_1, \ldots, a_{p+1}$ and (θ, Θ).

The component v_t is a linear function of $a_{p+2}, \ldots, a_t$ with coefficients determined by (θ, Θ). When $\Theta = 1$, then $(1 - B)v_t = (1 - \theta B)a_t$, and the seasonal component is s_t, i.e., it is stable (periodic). Further, v_t is a_t if $\theta = 1$; but if $\theta = 0$, v_t is the random walk $\sum_{j=p+2}^{t} a_j$. The closer θ and Θ are to one, the more linear the visible trend in data from (3.1) is likely to be, and the more stable the visible seasonality. Findley and Martin (2003) provide formulas for v_t for other extreme cases in which θ and Θ have the values zero or one. When (θ, Θ) is close to (1, 0), these formulas indicate that v_t behaves effectively like a seasonal random walk within each calendar period (calendar month if $p = 12$) in a way that is independent for each of the p periods, so the evolution of the observed seasonal pattern will be quite erratic. When (θ, Θ) is close to (0, 0), the movement is even more complex, with the variance of v_t increasing as the cube of the series length divided by p. Diagnostics of some AMB filters from models with $\Theta = 0$ are presented in Subsections 3.2.3–4 and in A.3.

In practice, small estimated values of Θ, especially for moderate T, are associated with very erratic seasonal movements that can obscure trend movements. In this case, one expects the seasonal frequency components to have power at frequencies spread across broad intervals around seasonal frequencies (see Bloomfield 2000, pp. 81–82). Hence one expects the seasonal adjustment filter squared gains to have broad troughs at seasonal frequencies to suppress these components. This will be seen to happen for small Θ in the figures below (more clearly with larger T), just as narrow troughs will be seen when Θ is close to one, when the periodic component s_t in (3.2), whose spectrum is concentrated at the seasonal frequencies, becomes dominant.

Our method of obtaining the SEATS and X-11/12-ARIMA filter coefficients $c_{s,t}^T$ in (1.1) is detailed in Appendix B. From the $c_{s,t}^T$, we calculated and plotted the squared gain and phase delay function of the concurrent adjustment filters and the central adjustment filters for series of lengths 37, 49, 61 and 109 months associated with the airline model coefficient pairs (θ, Θ) with $\theta = 0.2, 0.4, 0.6, 0.8$ and $\Theta = 0.0, 0.4, 0.6, 0.8, 0.95$. For reasons of space, plots for fewer values of (θ, Θ) and mainly for lengths 49 and 109 are shown below. Cholette (1979) emphasized that seasonal adjustment of series shorter than five years should generally be avoided, if possible. Our results will further illuminate why this is so. Thus the length 49 filters that we analyze should be regarded as somewhat extreme. In fact, the plots for length 61 months, which are not presented, are less erratic than the plots provided for length 49 in some frequency intervals but more erratic in others and therefore not convincingly better. The plots for length 37 are more erratic than those for $T = 49$ except for the example with $\Theta = 0$ discussed in Subsection 3.2.4, whose gain and phase delay properties are nevertheless problematic.

X-11/12-ARIMA includes options that allow the user to specify various X-11 seasonal and trend filters. For the plots that include squared gains or phase delays of X-11/12-ARIMA filters, we always specified the 13-term Henderson trend filter. Except in Figures 1 and 2, we always used the *x11default* option of X-12-ARIMA, which specifies a 3×3 seasonal filter for the initial seasonal factor estimates and the 3×5 seasonal filter thereafter (see Ladiray and Quenneville 2001, which also describes how the program replaces any specified filter when the series is too short for its use; replacement filters were needed for all finite lengths T of this article.) For simplicity, we refer hereafter to the resulting seasonal adjustment filter, and to its modifications by model forecasts and backcasts, as the X-11/12-ARIMA default filter for length T. The *x11default* specification is the most frequent choice among the nine alternatives of the default automatic filter selection procedure of X-11/12-ARIMA.

3.1. *Diagnostics of Automatically Chosen X-11/12-ARIMA Seasonal Adjustment Filters Can Be Close to Those of SEATS Filters for a Variety of Airline Models*

As mentioned, our focus is on SEATS filters, not on their comparison with X-11/12-ARIMA filters, which play an auxiliary role in the article. Only in Figures 1 and 2 do we show diagnostics from comparable X-11/12-ARIMA and SEATS filters, mainly to illustrate that some of our conclusions apply to filters from both approaches. The X-11/12-ARIMA filters in Figures 1 and 2 are filters we have seen the programs automatically choose for series with estimated airline model coefficients close to those of the figures. (They were not chosen to be optimally close to the SEATS filters in any sense.)

Figure 1 contains plots of the squared gain function for series length $T = 109$ of concurrent (left side of figure) and symmetric (right side of figure) SEATS and X-11/12-ARIMA filters resulting from a model with $\theta = 0.6$ and either $\Theta = 0.4$ (top), $\Theta = 0.6$ (middle) or $\Theta = 0.8$ (bottom), corresponding to rather erratic, average, and rather smooth seasonal movements, respectively. The default filter described above was specified for the middle plots, whereas alternative filters that can be specified automatically by X-11/12-ARIMA appear in the top and bottom plots, with those on top involving the 3×3 seasonal filter specification and those in the bottom plots the 3×9 specification. The phenomenon that the squared gain troughs around the seasonal frequencies, $\lambda = 1, 2, \ldots, 6$ cycles/year, are deeper for the symmetric filters than for the concurrent filters is a general one, explanations for which are given for SEATS filters in Subsection 3.2.1. So is the decrease, for fixed T, in the size of the largest oscillations as Θ decreases, also visible in the phase delay plots of Figure 2, (see Subsection 3.2.2.)

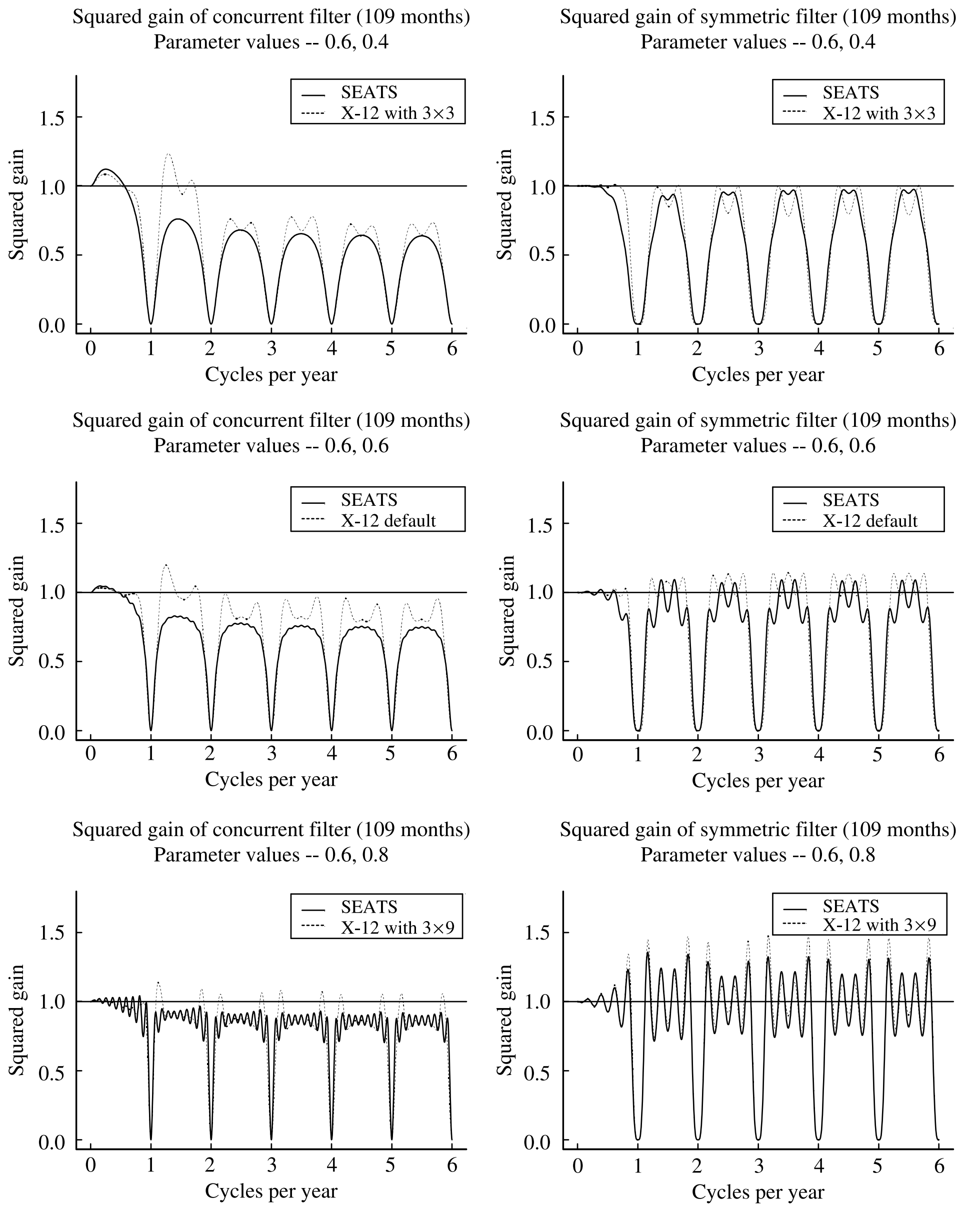

Fig. 1. SEATS and X-11/12-ARIMA squared gain functions of concurrent (left) and symmetric (right) filters for $\theta = 0.6$ *and* $\Theta = 0.4, 0.6, 0.8$. *The default X-11/12-ARIMA filter's squared gains appear in the middle graphs. Those of other automatic choices of X-11/12-ARIMA filters appear in the top and bottom graphs. In each graph, the squared gains of the X-11/12-ARIMA filters resemble those of SEATS filters*

3.2. Properties of SEATS' Squared Gain and Phase Delay Functions with Reference to Default X-11/12-ARIMA Functions for Various θ and Θ

We now compare properties of SEATS filters for different models and series lengths, often using the more fixed properties of the default X-11/12-ARIMA filters as a reference in the

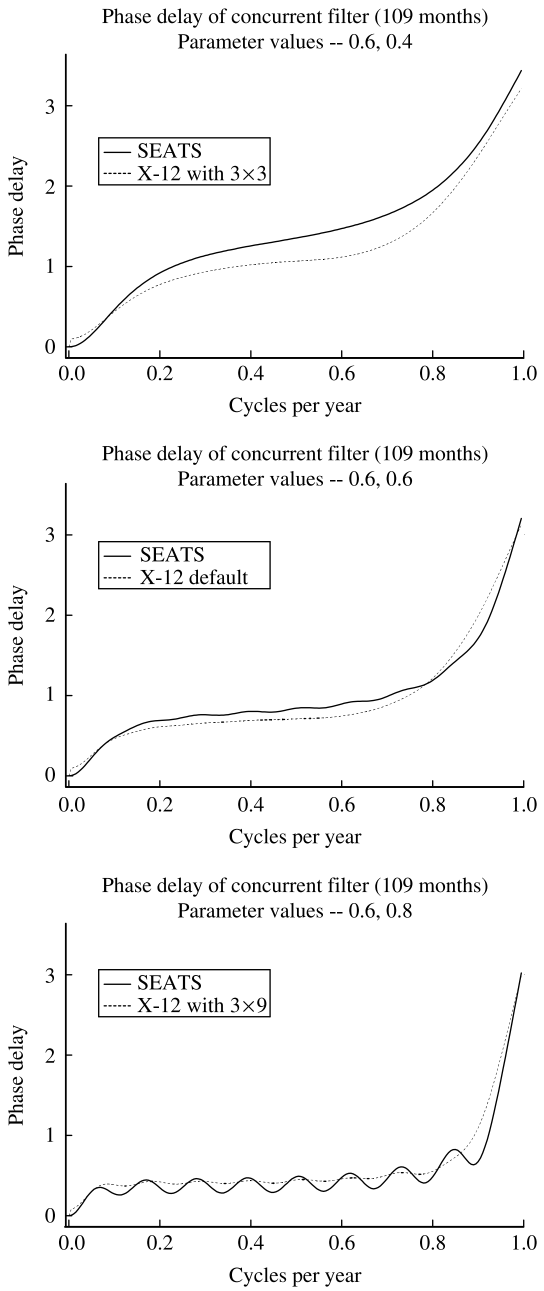

Fig. 2. SEATS and X-11/12-ARIMA concurrent phase delay functions for $\theta = 0.6$ *and* $\Theta = 0.4, 0.6, 0.8$. *The default X-11/12-ARIMA filter's phase delay appears in the middle graphs. Those of other automatic choices of X-11/12-ARIMA filters appear in the top and bottom graphs. In each graph, the phase delays of the X-11/12-ARIMA filters resemble those of SEATS filters*

graphs to make differences between the properties of different SEATS filters easier to see. Additional graphs illustrating the phenomena we mention can be found in Findley and Martin (2003).

3.2.1. Squared Gain Functions

Squared gain graphs for length $T = 49$ and $\theta = 0.2, 0.6, 0.8$ are shown in Figure 3 ($\Theta = 0.4$), and Figure 4 ($\Theta = 0.8$). After identifying visual features of interest, we offer analytic explanations for some of them. One thing to keep in mind is that, from the point of view of interpretation, squared gain values larger than one suggest that the corresponding frequency component is stronger in the estimate $\hat{N}_t$ than in the component N_t, at least in the sense of contributing more variability. This is especially relevant for the large peaks associated with $T = 49$. By contrast, squared gain values less than one are not always problematic. See Subsection A.2 for a background discussion of these issues.

Effects of Increasing $\boldsymbol{\Theta}$

Figure 3 shows that when $\Theta = 0.4$, the impact of θ is greater on the concurrent filters than on the symmetric filters and greater on the SEATS filters than on the X-11/12-ARIMA filters, as would be expected since, for the latter, it is only the forecast and backcast functions that change with θ. For the concurrent filter squared gains, much more than for symmetric filter squared gains, over the intervals between the seasonal frequencies, as θ increases, the average squared gain level decreases, indicating greater suppression of frequency components on average for components with periods shorter than a year ($\lambda > 1$). This suppression becomes less pronounced as Θ increases, being modest when $\Theta = 0.8$: compare the bottom graphs of Figure 3 for $(\theta, \Theta) = (0.8, 0.4)$ with the bottom graphs of Figure 4 for $(\theta, \Theta) = (0.8, 0.8)$.

Effects of Increasing $\boldsymbol{\Theta}$

Comparing the graphs in each row of Figure 3 with the corresponding graphs in Figure 4, and examining Figure 1 progressing from top to bottom, one sees that as Θ increases, resulting in an increasing contribution of the periodic component s_t to the seasonal component as discussed below (3.2), the troughs at seasonal frequencies of the SEATS squared gain functions of the concurrent filters become narrower and so do the sizes of the largest oscillations of the SEATS functions, a phenomenon we discuss in Subsection 3.2.2. Also, $\max_{0 \leq \lambda < 1} G^2(\lambda)$, the maximum amplification over trend and cycle frequencies, usually increases with Θ.

Effects of Filter Length T

Compare the second row of Figure 3 with the first row of Figure 1 ($(\theta, \Theta) = (0.6, 0.4)$) and the second row of Figure 4 with the third row of Figure 1 ($(\theta, \Theta) = (0.6, 0.8)$). The shorter filters have larger oscillations than their counterparts in Figure 1. The troughs at the seasonal zeros of the symmetric filters are broader for the shorter filters.

The broad troughs are just one example of the tendency of the squared gains of the symmetric filters for short series to have rapid changes of amplitude over intervals away from the seasonal frequencies, i.e., away from the frequencies of the components targeted for suppression. When they are not near seasonal frequencies, such changes seem undesirable

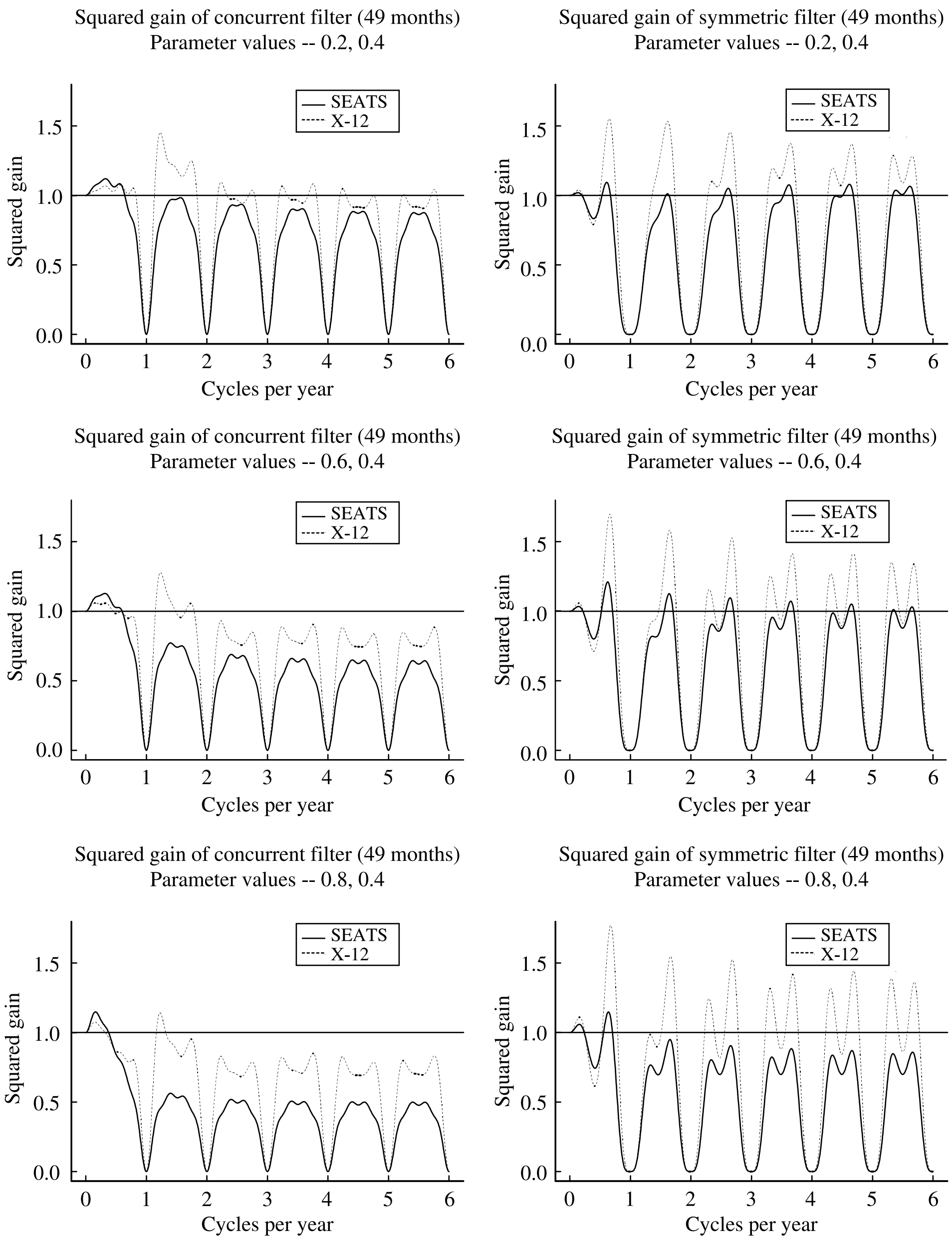

Fig. 3. SEATS and X-11/12-ARIMA squared gain functions for $\Theta = 0.4$ *and series of length 49. From top to bottom,* $\theta = 0.2, 0.6, 0.8$. *The impact of* θ *is greater on the concurrent filters (left) than on the symmetric filters (right), and greater on SEATS than on X-11/12-ARIMA, as would be expected since only the contribution of the forecasts is changing for the latter*

because they indicate that some nonseasonal frequency components of the data at frequencies close to one another will be treated substantially differently by the filter. Such an "inconsistent" treatment of neighboring frequency components could, if these components are of significant size in the data, make certain aspects of the seasonally adjusted series

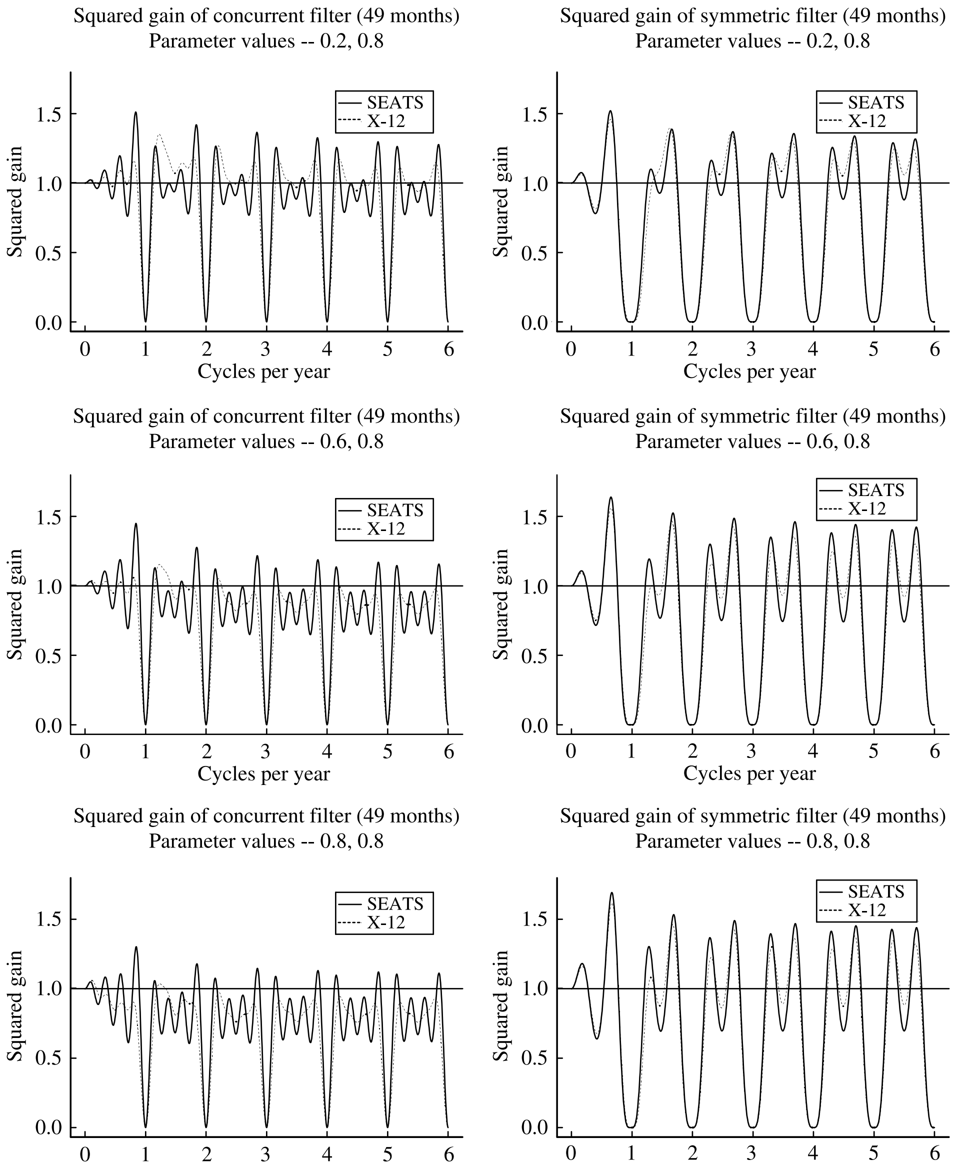

Fig. 4. SEATS and X-11/12-ARIMA squared gain functions for $\Theta = 0.8$ *and series of length 49. From top to bottom,* $\theta = 0.2, 0.6, 0.8$. *These are rather similar to Figure. 3 with SEATS concurrent functions even closer to one away from seasonal frequencies, while being more oscillatory. Note that here as in Figures 1, 3, 6, 8, and 10, the concurrent filters' squared gain troughs around seasonal frequencies are narrower than those of the symmetric filters*

misleading or, at the very least, susceptible to large revisions when additional data enable the use of longer filters with more stable squared gains. The greater tendency of the squared gains to have values larger than one for larger values of Θ can also cause problems regarding interpretation, (see Subsection A.2), and so is a further reason to consider series length 49 to be quite short for seasonal adjustment, in agreement with Cholette (1979).

Zeros at Seasonal Frequencies and Associated Troughs

We now provide analytic explanations of the phenomenon that, for a given pair of model coefficients, the symmetric filter's squared gain has broader (deeper) troughs at the seasonal frequencies than the squared gain of the concurrent filter. For infinite filters, this phenomenon (visible in Figures 6 and 7) can be understood by noting that the factor $|\delta_S(e^{-i(2\pi/12)\lambda})|^2$ in the final formula of (A.3) for the transfer function of the symmetric bi-infinite seasonal filter is replaced by the factor $\delta_S(e^{-i(2\pi/12)\lambda})$ in the formula (15) of Bell and Martin (2004) for the transfer function of the infinite asymmetric filters, and that $\delta_S(B) = \sum_{j=0}^{11} B^j$ for seasonal adjustment filters of the airline model. Hence, the zeros of the squared gains of symmetric filters at seasonal frequencies have order (multiplicity) four, whereas those of the asymmetric filters have order two, with the result that squared gains of the symmetric filter must have smaller amplitudes near these frequencies, and therefore broader troughs.

Appendix C shows that the zeros at seasonal frequencies of the transfer functions of all the symmetric and asymmetric finite AMB filters are of order (at least) one. For the odd-length symmetric filters, it is further shown that the zero at $\lambda = 6$ always has order two. However, limited numerical results (not presented) indicate that, for symmetric filters for monthly data, there is always a k such that $e^{-i(2\pi/12)k}$ is a zero of $C(z) = \sum_{j=-\tau}^{\tau} c_j z^j$ of order one. Interestingly, such a zero was always close to another zero of this transform, i.e., was "almost" a zero of order two. For concurrent filters, each $e^{-i(2\pi/12)k}$ was an isolated zero of order one.

Remark 3.1 Because of its narrower squared gain troughs at seasonal frequencies, one might expect the concurrent filter to be less able than the symmetric filter to track a variable seasonal pattern to such an extent that the concurrent adjustment could have a nonnegligible seasonal component. But for long enough series from a sufficiently good model, this does not happen. More precisely, when the time series conforms to the seasonal ARIMA model used to obtain the filters, Bell (1995) shows that, under the assumption of Appendix D below, the series of concurrent model-based adjustments from the infinite past has a *nonseasonal* ARIMA model.

3.2.2. Sources of Oscillatory and Related Behavior

For fixed Θ, the sizes of the largest oscillations decrease with increasing T (see Figures 1,3, and 4) and vanish when $T = \infty$ (see Subsection 3.2.4 and Figure 6). The fact that, for fixed T, they decrease as Θ decreases can be explained by the fact that the filter coefficients decay more rapidly for smaller Θ, with the result that the filters resemble more the filters with larger T (see Figure 7). Although the infinite filters of our examples have transfer functions that are continuous, the oscillatory behavior of the associated finite-filter transfer functions resembles the well-known Gibbs phenomenon that is associated with the approximations to a discontinuous transfer function. However, we are not able to make quantitative statements about the oscillations of squared gains of seasonal adjustment filters like those available for the Gibbs phenomenon (see pp. 112–114 of Bloomfield 2000). We can offer a qualitative conjecture concerning the phenomenon that, for a given model coefficient pair and series length T, the symmetric filter's squared gain has larger oscillations than the squared gain of the concurrent filter. Two sources for this phenomenon suggest themselves. First, the many symmetry constraints on the symmetric filter's coefficients restrict the flexibility of its gain function: the symmetric filters have only $(T+1)/2$ distinct coefficients, whereas the concurrent filters have T. Second, the symmetric filter squared gains have amplitudes between seasonal frequencies that are farther from zero on average, resulting in a persistently larger range of movement. With fewer distinct coefficients than the concurrent filter, this larger range must necessitate larger ancillary movements.

3.2.3. Phase Delay Functions

Figures 2 and 5–10 show concurrent filter phase delay functions over the frequency interval $0 < \lambda < 1$. As we noted, this interval includes the frequencies usually associated with trends and business cycles. Beyond this interval, the continuous phase delays tend to the value 5.5 as $\lambda \rightarrow 6$ (see Appendix D and Figure 11). For small values of T, Figures 5 and 6 show that finite concurrent filter phase delay functions can have large oscillations that show the possibility of somewhat inconsistent delays of neighboring frequency components. Oscillations, if they occur, are smaller for $T = 109$ than for $T = 49$, but apart from this difference, for a given pair of coefficient values, the basic features of the phase delays are the same for both series lengths. For the SEATS filters, except at the lowest trend frequencies, it can be seen that, for a given θ, phase delay increases as Θ decreases (see Figures 2, 5 and 7). Also, for a given Θ, phase delay increases as θ increases: compare the successive graphs in Figure 2 (where $\theta = 0.6$) with those of the second column of Figure 5 (where $\theta = 0.8$). Although the phase delay value at a given frequency can be influenced by gain function values at distant frequencies (see Appendix D), phase delay is often greatest near where the squared gain values in $0 \leq \lambda < 1$ are smallest, i.e., where frequency component suppression is greatest: compare the squared gains of the first column of Figure 1 with the corresponding phase delays of Figure 2, and the squared gain pairs of the second rows of Figures 6 and 8 with the phase delay pairs of the third rows. Also compare the SEATS and X-12-ARIMA functions for identical values of θ and Θ in Figures 1,2,3, and 5.

We conclude from these observations that greater smoothing (suppression) usually leads to greater phase delay of components associated with business cycle frequencies. The smaller airline model phase delays at higher frequencies in Figure 10 (which is discussed in Subsection A.3) appear to be an exception, but this might be due to the fact that its filter does substantially less suppression at frequencies $\lambda > 2$. A further analysis of the graphs supports the observation of M. Wildi (personal communication) that larger amplifications at frequencies close to zero usually leads to smaller phase delays. The phase delay graphs of Figure 8 (which is discussed in Section 4) over frequencies roughly between 0.1 and 0.4 appear as an exception, but the explanation may lie in the fact that the airline model squared gain does much less suppression at most frequencies $\lambda > 1$.

3.2.4 Comparison of Finite and Infinite Length Concurrent and Symmetric Filter Functions

In Figure 6, squared gain and phase delay plots are shown for filters of length 109 (on the left) and infinite length (on the right) for airline models with $\theta = 0.8$ and $\Theta = 0.8, 0.95$. One sees from Figure 6 and from the third row of Figure 4 ($(\theta, \Theta) = (0.8, 0.8)$) that

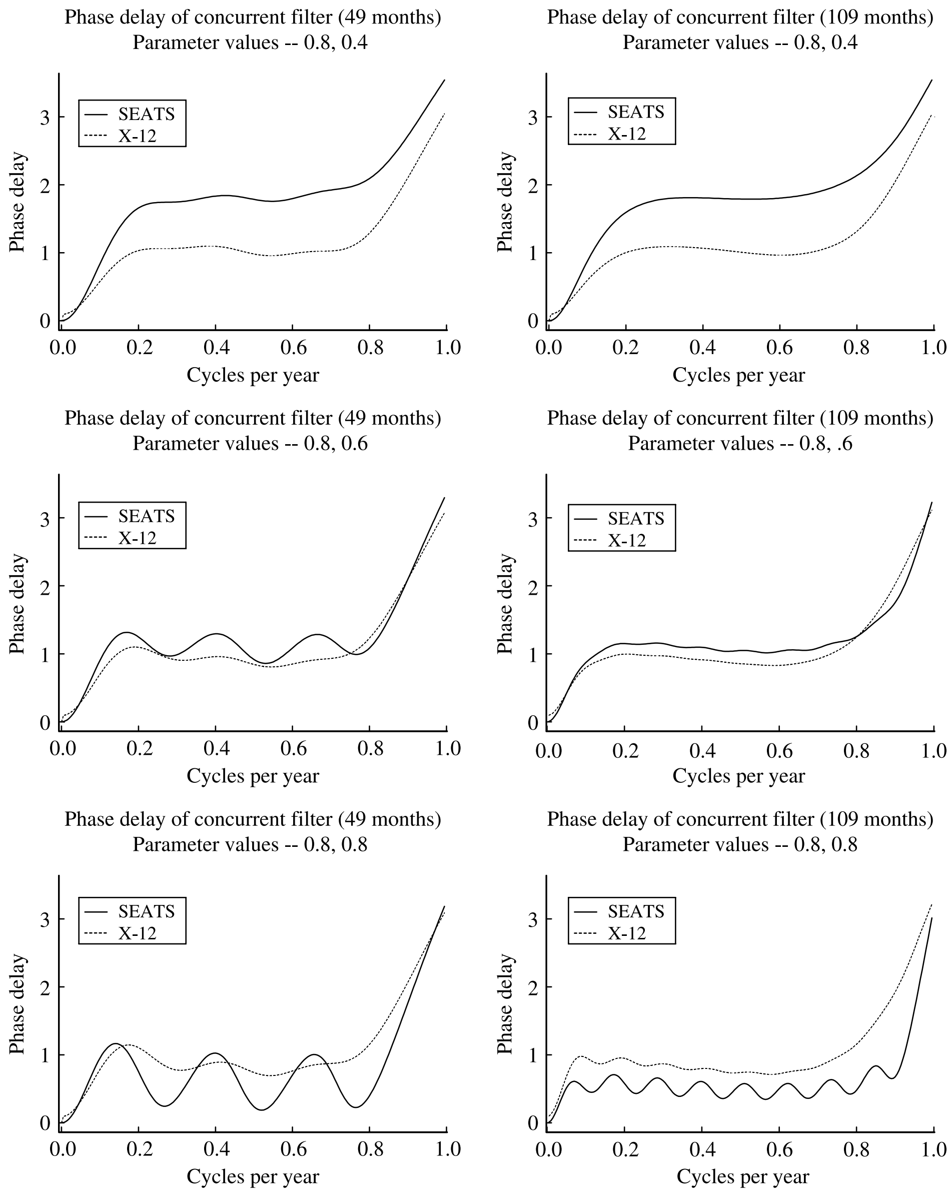

Fig. 5. SEATS and X-11/12-ARIMA concurrent filter phase delay functions for $\theta = 0.8$. *From top to bottom,* $\Theta = 0.4, 0.6, 0.8$ *for series lengths 49 (left) and 109 (right)*

squared gains of the infinite filters often give no indication of whether or where the finite filters will have broad troughs, rapid movements or oscillations. The squared gains of the infinite filters never have rapid oscillations like those of the finite filters. Similarly, the infinite-filter phase delays are smooth and offer no information about the location or size of oscillations in the finite-filter phase delay functions, but they do seem to suggest where the phase delays of the finite filters will be large.

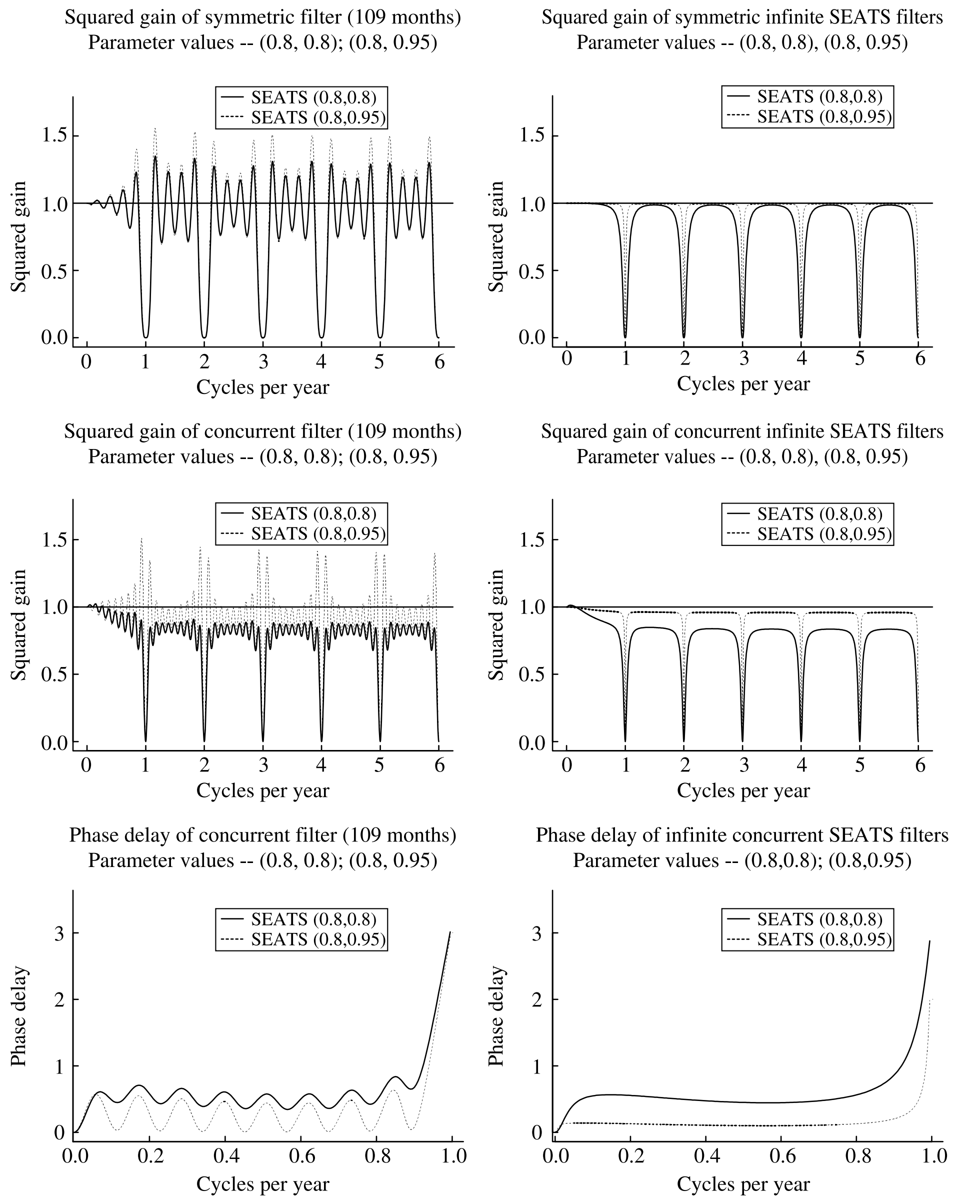

Fig. 6. Squared gain and phase delay functions for SEATS filters of length 109 (left) and infinite length (right) for models with $\theta = 0.8$ *and* $\Theta = 0.8, 0.95$. *For the symmetric filters, the infinite filters' squared gains give little indication of shapes of the finite filters' squared gains, which differ little. For all of the finite-filter functions, the larger oscillations of the* $\Theta = 0.95$ *functions make their filters seem less desirable than the filters of the model with* $\Theta = 0.8$

Turning to the differences between the graphs for $\Theta = 0.8$ and $\Theta = 0.95$ in Figure 6, although the infinite symmetric filter squared gains are quite distinct, the finite symmetric filter squared gains are very similar and rather different from the infinite filter functions. Indeed, in several ways, the latter are not very indicative of the former. The finite-filter

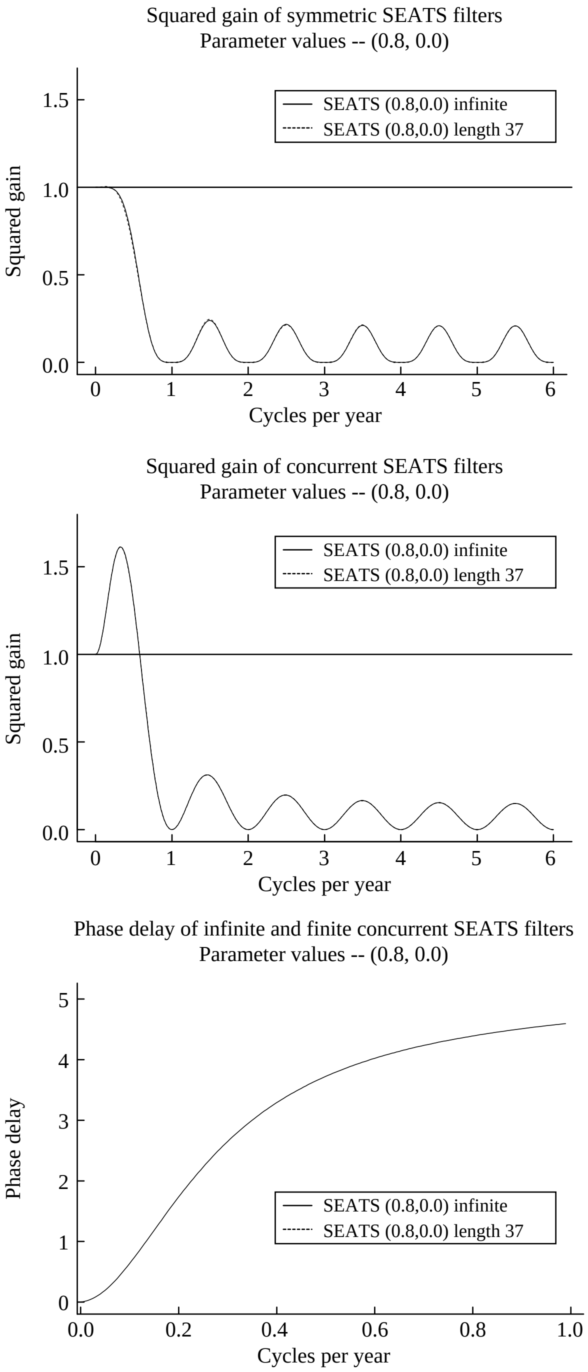

Fig. 7. Squared gain and phase delay functions of infinite and finite (length 37) SEATS filters with $\theta = 0.8$ and $\Theta = 0.0$. The spectral functions are almost indistinguishable because of the rapid decay of the coefficients of the infinite filter

symmetric squared gains have almost identical troughs at the seasonal frequencies, and the slight narrowing at the tops of the troughs for $\Theta = 0.95$ comes at the cost of more extended rapid movements throughout the graph. Considering this feature of the $\Theta = 0.95$ example together with the very rapid oscillations of its concurrent squared gain, it appears that, when parameter estimates around $\theta = 0.8$ and $\Theta = 0.95$ are obtained for a series of length 109, it might be preferable to use $\Theta = 0.8$ filters to obtain the seasonal adjustment.

These observations demonstrate important inadequacies of the squared gains of the infinite filters as diagnostics. They cannot generally be used to predict which trend or cyclical components of the observed series that are of possible interest to the data user are likely to be amplified or treated somewhat inconsistently by the finite length filters. Similarly, the infinite filter squared gains should generally not be used to make inferences about differences between X-11/12-ARIMA and SEATS adjustments, because these are always produced by finite filters.

There are exceptions. For example, when $(\theta, \Theta) = (0.8, 0.0)$ the filter coefficients (not shown) decay so rapidly that there is essentially no difference in Figure 7 between the infinite filter functions and those from length $T = 37$. These filters strongly suppress frequency components with $\lambda > 1$ but the large peaks of the values $G^2(\lambda) > 1$ roughly centered about $\lambda = 1/3$ show that the concurrent filters can strongly exaggerate business cycle components. This property, and the large phase delays, indicate that seasonal adjustments via models with $\Theta = 0$ may have limited value for current economic analyses (see also Subsection A.3 below).

4. Applying the Diagnostics to Choose Between AMB Filters from Competitive Models

We now consider a series fit almost equally well by two models whose seasonal adjustment filters have important differences that are revealed by the squared gain and phase delay functions.

The series X41020 of monthly U.S. Exports of Cookware, Cutlery, House and Gardenware from January 1989 through November 2001 is one for which the BIC model selection criterion of Schwarz (1978), used by TRAMO to select models for SEATS, prefers the 1-12-13 model of Findley, Martin, and Wills (2002),

$$(1 - B)(1 - B^{12})Z_t = (1 - \theta_1 B - \theta_{12} B^{12} - \theta_{13} B^{13})a_t \tag{4.1}$$

with $BIC = 4803.96$, over the airline Model (3.1), which imposes the constraint $\theta_{13} = -\theta_1\theta_{12}$ and has $BIC = 4804.73$. For this series Z_t is the log of the trading day and outlier adjusted data. The parameter estimates from X-12-ARIMA are $\theta_1 = 0.3569$, $\theta_{12} = 0.6589$, $\theta_{13} = -0.4245$ for Model (4.1) and $\theta = 0.4182$, $\Theta = 0.6351$ for Model (3.1). The Ljung-Box Q statistic at lag 24 has the p-values 0.275 for Model (4.1) and 0.304 for Model (3.1), so both models are acceptable by the criteria of TRAMO. Within the lag range $l < 24$, there are three lags, $l = 6, 9, 10$, for which the Model (4.1) has Q's with p-values less than 0.05 (the respective p-values are 0.041, 0.043 and 0.037), something that does not occur for Model (3.1), a fact that could lead some modelers to prefer Model (3.1) for this series. The models are therefore competitive.

Figure 8 shows that their seasonal adjustment filters have quite different frequency domain characteristics. The concurrent phase delays are comparable for long-term trend movements associated with quite small λ, but for Model (4.1) the phase delay becomes more than one month for any cyclical component whose period is shorter than five years, in contrast to the phase delay of Model (3.1). The squared gain functions of Model (4.1) show much more smoothing of higher frequency components and short-term business cycle components. The concurrent filter of Model (4.1) has a greater tendency to emphasize, and perhaps exaggerate, some long-period components. Given the information provided by the diagnostics of Figure 8, we would expect few seasonal adjusters to be indifferent about which of these two models was used for seasonal adjustment and, also, that a number of seasonal adjusters would prefer seasonal adjustments from Model (3.1), even though they will be less smooth than those from the Model (4.1) preferred by BIC. (Although smoothness is not an intrinsic quality of a good seasonal adjustment, among two adjustments of comparable credibility, the smoother one is often preferred in the hope that it will be more interpretable, usually with no recognition of the possible consequences of greater phase delay usually associated with greater smoothing.) Thus it is important that the frequency domain diagnostics discussed in this article be made available to users of model-based seasonal adjustment software. Additional examples of competitive model pairs whose seasonal adjustment filters have substantially different squared gain and phase delay properties are presented in Findley, Martin, and Wills (2002).

5. Trend Filters

An analogous analysis of model-based trend filters, which suppress seasonal and irregular components, can be expected to yield squared gains with features qualitatively similar to those seen in Figures 1–7, except that the squared gain values will be much closer to zero from some point near the first seasonal frequency on and the phase delays can be much greater (see Figure 9 for the case $(\theta, \Theta) = (0.8, 0.4)$ with length 109). The phase delay of the concurrent trend filter in Figure 9 is essentially twice that of the concurrent seasonal adjustment filter throughout most of $0 < \lambda < 1$. When $(\theta, \Theta) = (0.6, 0.8)$, the concurrent trend filter phase delay (not shown) is mostly more than three times that seen in Figure 2 for the seasonal adjustment filter. The trend filter gain and phase graphs of Dagum and Luati (2002) and the trend time shift graphs of Wildi (2004) offer further illustrations that, when one trend filter effects greater smoothing than another, it will usually also have greater phase delay.

6. Concluding Remarks

Our main findings are the following. (1) The squared gains of the infinite AMB filters are not reliable diagnostics for series of the lengths considered in this article. They generally fail to show where the squared gains of the finite filters actually used are larger than one and where they have large and rapid changes away from seasonal frequencies, information that can help to avoid misinterpretations of the adjusted series. (2) The squared gains and phase delays of the concurrent adjustment filters provide information that is different from

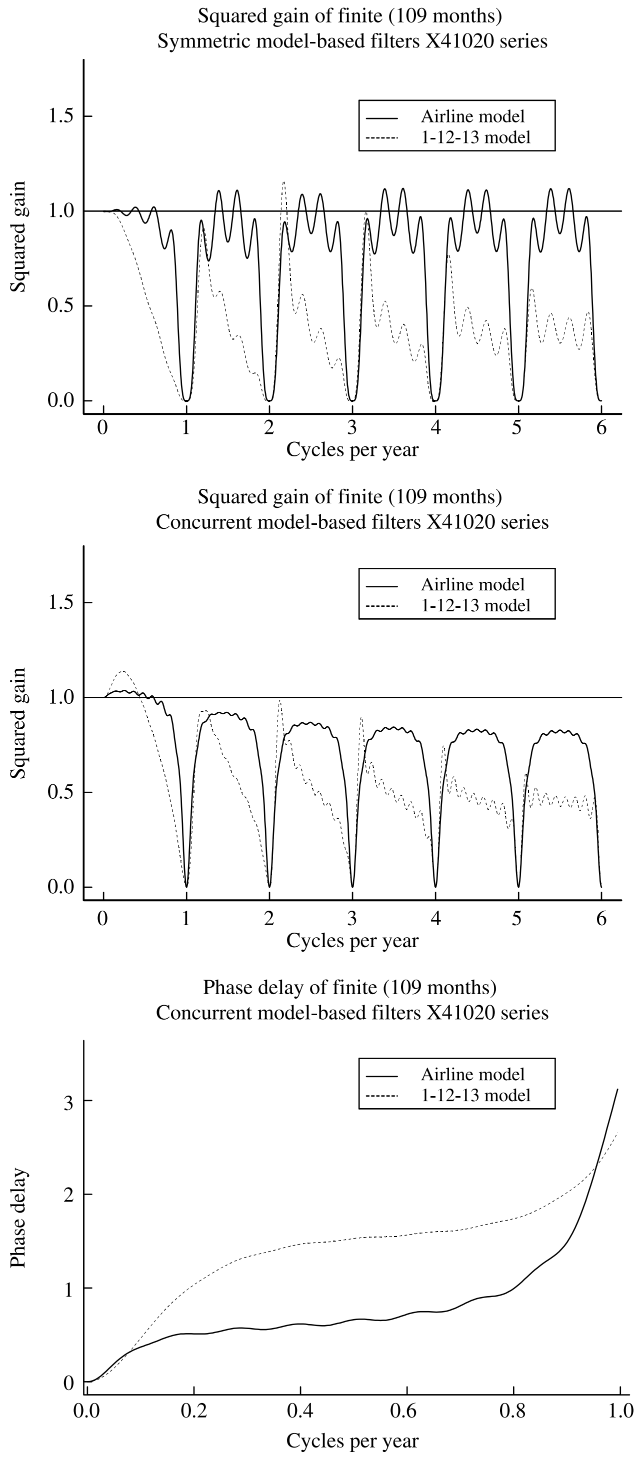

Fig. 8. Squared gains and phase delays of filters of length 109 for the Models (3.1) (solid line) and (4.1) (dashed line) for an Exports series. Although the models have similarly good fits to the data, the plots reveal that their seasonal adjustment filters have substantially different properties

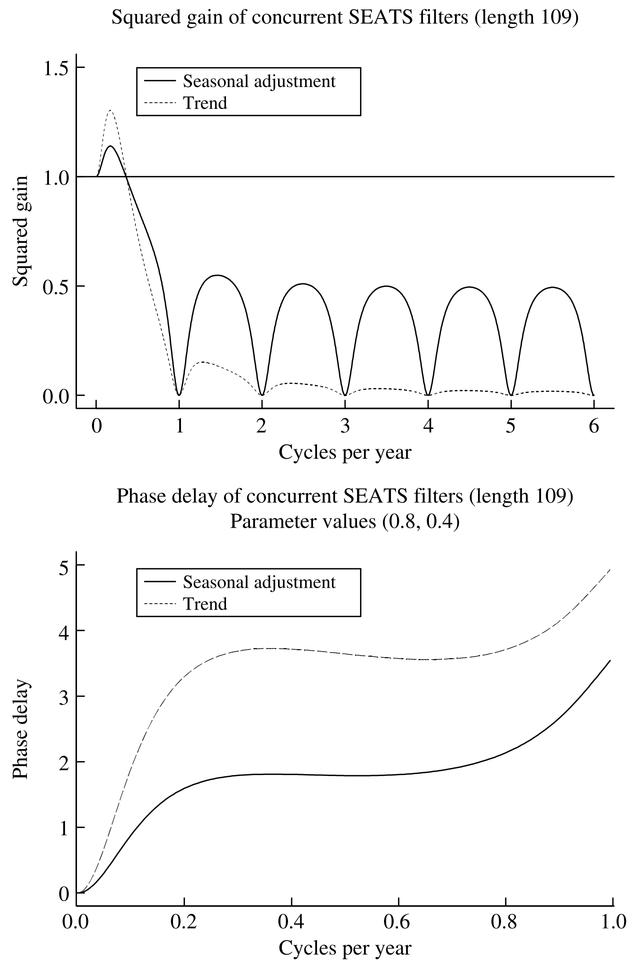

Fig. 9. Squared gains and phase delays of concurrent SEATS seasonal adjustment and trend filters of length 109 for (3.1) with $\theta = 0.8$ *and* $\Theta = 0.4$. *Note that the phase delay of the trend filter is roughly twice that of the seasonal adjustment filter. Consequently the trend could have large timing distortions for many business cycle components*

(and more valuable for real-time analyses than) that provided by the squared gains of the symmetric filters. (3) The phase delays of concurrent filters indicate that the extent to which concurrent filters delay business cycle information increases with increased smoothing, in a way that depends more strongly on the seasonal than on nonseasonal moving average parameters, perhaps because the former usually have more influence on the decay rate of filter coefficients. (4) Trend filters can have much larger phase delays than seasonal adjustment filters. They illustrate the phenomenon that greater smoothing usually results in greater phase delay. (5) For short series, with roughly 60 observations or less, the squared gain analyses indicate that seasonal adjustment is likely to be problematic. Except with series whose frequency components are mostly restricted to short frequency intervals around trend and seasonal frequencies, e.g., series with nearly stable seasonals, almost linear trends and quite moderate irregularity, interpretation of the seasonally adjusted series will be difficult. (6) Airline models with $\Theta = 0$ produce series with very erratic seasonals and very short AMB seasonal adjustment filters that effect extreme suppression with very large phase delays (see Figure 7). Hence they result in model-based seasonal adjustments likely to be quite difficult to interpret reliably. These models are used by SEATS as an (optionally) automatic alternative to inadmissible models and were used in the study of X-12-ARIMA of Matas Mir and Rondonotti (2003) to illustrate performance with changing seasonality.

With appropriate reinterpretations of Conclusions (3) and (6), whose formulations depend in part on the parameterization of ARIMA-models, one can expect our conclusions to apply to the seasonal adjustment filters of other model-based signal extraction methods, e.g., those of BAYSEA (Akaike and Ishiguro 1980, based on Akaike 1980), DECOMP (Kitagawa 1985, based on Kitagawa 1981) and STAMP (Koopman, Harvey, Doornik, and Shepherd 1995, based on Harvey 1989).

Appendix A: Perspectives for Evaluating Squared Gains and Phase Delays

A.1. Pseudo-spectral Generalizations of Spectral Density Identities

In the case in which Z_t is a zero-mean covariance stationary time series having the decomposition $Z_t = S_t + N_t$, where S_t and N_t are uncorrelated series with spectral densities $f_S(\lambda)$ and $f_N(\lambda)$ respectively, the spectral density of Z_t has the decomposition

$$f_Z(\lambda) = f_S(\lambda) + f_N(\lambda) \tag{A.1}$$

We assume that $f_Z(\lambda) > 0$ for all λ. At frequencies at which $f_N(\lambda) > 0$, it follows from (2.1) that for any estimator $\hat{N}_t = \sum_j c_j Z_{t-j}$, model-based or not, with $G(\lambda) = |\sum_j c_j e^{-i(2\pi/12)j\lambda}|$, we have

$$f_{\hat{N}}(\lambda) = \left\{ G^2(\lambda) \frac{f_Z(\lambda)}{f_N(\lambda)} \right\} f_N(\lambda) \tag{A.2}$$

In the nonstationary case, if Z_t has an ARIMA model, there is a generalization of the argument leading to (A.2) that requires the components S_t and N_t to follow ARIMA models whose differencing polynomials $\delta_S(B)$ and $\delta_N(B)$ have no common zero (as happens with $\delta_S(B) = 1 + B + \cdots + B^{p-1}$ and $\delta_N(B) = (1-B)^2$ in the airline model (3.1)). Then $\delta_Z(B) = \delta_S(B)\delta_N(B)$ transforms Z_t into a stationary ARMA process $z_t = \delta_Z(B)Z_t$ whose spectral density $f_z(\lambda)$ is assumed to be strictly positive. Concerning the spectral densities $f_s(\lambda)$ and $f_n(\lambda)$ of the respective ARMA processes $s_t = \delta_S(B)S_t$ and $n_t = \delta_N(B)N_t$, we further assume that when $f_s(\lambda)$ or $f_n(\lambda)$ is zero for some λ_0, the associated value of its differencing polynomial, $\delta_S(e^{-i(2\pi/12)\lambda_0})$ or $\delta_N(e^{-i(2\pi/12)\lambda_0})$, respectively, is nonzero. Then (A.1) holds for what are called the pseudo-spectral density functions (Hillmer and Tiao, 1982), $f_Z(\lambda) = f_z(\lambda)/|\delta_Z(e^{i(2\pi/12)\lambda})|^2$, $f_S(\lambda) = f_s(\lambda)/|\delta_S(e^{i(2\pi/12)\lambda})|^2$, and $f_N(\lambda) = f_n(\lambda)/|\delta_N(e^{i(2\pi/12)\lambda})|^2$, and so does (2.1) in the sense that correct results are obtained for $z_t = \delta_Z(B)Z_t$. The pseudo-spectral density version of (A.2) follows from these generalizations of (A.1) and (2.1). We note that

$$\frac{f_N(\lambda)}{f_Z(\lambda)} = \frac{\left|\delta_N\left(e^{-i(2\pi/12)\lambda}\right)\right|^{-2} f_n(\lambda)}{\left|\delta_S\left(e^{-i(2\pi/12)\lambda}\right)\right|^{-2} f_s(\lambda) + \left|\delta_N\left(e^{-i(2\pi/12)\lambda}\right)\right|^{-2} f_n(\lambda)}$$

$$= \frac{\left|\delta_S\left(e^{-i(2\pi/12)\lambda}\right)\right|^{2} f_n(\lambda)}{\left|\delta_N\left(e^{-i(2\pi/12)\lambda}\right)\right|^{2} f_s(\lambda) + \left|\delta_S\left(e^{-i(2\pi/12)\lambda}\right)\right|^{2} f_n(\lambda)} \quad \text{(A.3)}$$

As Bell (1984) shows, the function $G^{WK}(\lambda) = f_N(\lambda)/f_Z(\lambda)$ is the transfer function of a mean square optimal linear estimator of N_t of the form $\hat{N}_t = \sum_{j=-\infty}^{\infty} c_j Z_{t-j}$ minimizing the mean squared error $E(\hat{N}_t - N_t)^2$. Thus it is the pseudo-spectral density generalization of the Wiener-Kolmogorov transfer function for stationary Z_t given as (10.3.7) of Priestley (1981).

A.2. The Digital Signal Processing and Mean Square Optimal Filtering Perspectives

A.2.1. Competing Criteria

It was demonstrated in Section 4 that AMB filters from competitive models can have quite different squared gain and phase delay properties. In fact, the minimum mean square signal extraction error criterion that gives rise to the AMB filters does not insure that an AMB filter with smaller mean squared error will have smaller phase delay over trend and business cycles frequencies, or have gain function properties there that are likewise better from the perspective of interpretability of the filter output. For the purposes of seasonal adjustment and related analyses, the classical Digital Signal Processing (DSP) perspective of Rabiner and Gold (1975) and Oppenheim and Schafer (1975) generally favors filters supporting interpretability in the sense of having phase delays small enough for the user's purposes and gain properties that cause the filter output to have spectral properties like those of the signal being estimated, in the sense we now describe.

A.2.2. Interpretability and Squared Gains

When phase properties are not an issue, the implicit signal extraction ideal of the DSP perspective seems to be an estimator $\hat{N}_t = \sum_j c_j Z_{t-j}$ such that

$$f_{\hat{N}}(\lambda) = f_N(\lambda) \quad \text{(A.4)}$$

holds, at least over the frequency intervals of greatest interest (see Rabiner and Gold 1975 or Bloomfield 2000). As Wecker (1979) observed, and as follows from (A.2), (A.4) is achieved by any filter whose gain function is

$$G^{DSP}(\lambda) = G^{WK}(\lambda)^{1/2} \quad \text{(A.5)}$$

The functions $G^{DSP}(\lambda) = G^{WK}(\lambda)^{1/2}$ are not rational functions of $e^{i(2\pi/12)\lambda}$ and, for the necessarily infinite filters with this transfer function, no tractable finite approximations with some ideal property are known, so this ideal does not provide finite filters directly. Instead, it sometimes offers an ideal squared gain $G^{DSP}(\lambda)^2 = G^{WK}(\lambda)$ to which squared gains of competing finite filters can be compared to see which is closer to the ideal over the frequency interval (or intervals) of greatest interest. This requires a preferred model from which to obtain $G^{WK}(\lambda)$.

When the model choice is ambiguous, then the fact that every candidate model's $G^{WK}(\lambda)$ satisfies $G^{WK}(\lambda) \leq 1$ makes certain statements possible. First, it follows from (A.2) that whenever a filter defining an estimate $\hat{N}_t = \sum_j c_j Z_{t-j}$ is such that $G(\lambda) = |\sum_j c_j e^{-i(2\pi/12)j\lambda}| > 1$ over an interval where $G^{WK}(\lambda) > 0$, then $G^{WK}(\lambda)^{-1} \geq 1$ and (A.2) yield $f_{\hat{N}}(\lambda) > f_N(\lambda)$. Thus, $\hat{N}_t$ has larger variance in its components associated with these frequencies than the corresponding components of N_t and so could misrepresent frequency domain properties of N_t. Further, given two filters whose squared gains are such that $G^{(1)}(\lambda)^2 > G^{(2)}(\lambda)^2 > 1$ holds over an interval containing important frequencies for the filtered series, then the squared gain $G^{(2)}(\lambda)^2$ will be closer to any choice of $G^{DSP}(\lambda)^2 = G^{WK}(\lambda)$ over this interval, so its filter will be preferred (if the phase delay properties of both filters are acceptable).

A.3. An Application of $G^{DSP}(\lambda)^2 = G^{WK}(\lambda)$

We give an illustrative example using the series of sales volumes of large department stores (*Grands Magasins*) from January, 1990 through March, 2004 ($T = 171$) produced by the Chamber of Commerce and Industry of Paris (CCIP). J. Anas of CCIP communicated that CCIP found the SEATS seasonal adjustment obtained with TRAMO's automatically chosen (0, 1, 2) (0, 1, 0) model to be too smooth and used instead the adjustment from the estimated airline model, a model whose goodness-of-fit diagnostics are much worse but whose seasonal adjustment shows one or more known, nonseasonal, recent economic events of interest to CCIP not visible in the better-fitting (0, 1, 2) (0, 1, 0) model's adjustment.

Our fitted (0, 1, 2) (0, 1, 0) model[4] with $(\theta_1, \theta_2) = (1.0770, -0.3739)$, had $BIC = 907.76$ and very good modeling diagnostics, e.g., Ljung-Box Q-statistics whose p-values increased from a minimum of 0.118 at lag 2 to 0.899 at lag 24. By contrast, the fitted airline model, with $(\theta, \Theta) = (0.8169, 0.1454)$, had the much larger $BIC = 924.68$ and much worse modeling diagnostics, e.g., the Q-statistic at lag 24 was the first to have a p-value as large as 0.05. Therefore $G^{WK}(\lambda)$ of the (0, 1, 2) (0, 1, 0) model was used to define the DSP ideal squared gain.

Figure 10 shows overlay plots of this function with the squared gain functions of the two models' symmetric and concurrent filters of length 171. Our comparisons ignore frequencies at which the competing filters' square gains are visually very close. For the symmetric filter, this includes the interval $0 \leq \lambda < 1$. Over the remaining frequencies, the (0, 1, 2) (0, 1, 0) model's squared gain is closer to the DSP ideal somewhat more often than not. By contrast, for the concurrent filters, whose performance is more relevant for CCIP's concerns, the squared gain of the airline model is closer to the DSP ideal somewhat more often than not, especially over most of $0 \leq \lambda < 1$, a fact that supports CCIP's preference

[4] We used the same outlier regressor as CCIP, but did not have enough information to replicate the adjustments for French holidays used by CCIP. The only holiday regressor in our model is an Easter regressor automatically selected by X-12-ARIMA. Our use of different regressors could result in different ARIMA coefficient estimates and therefore different seasonal adjustment filters and squared gains, so our analyses might not apply to CCIP.

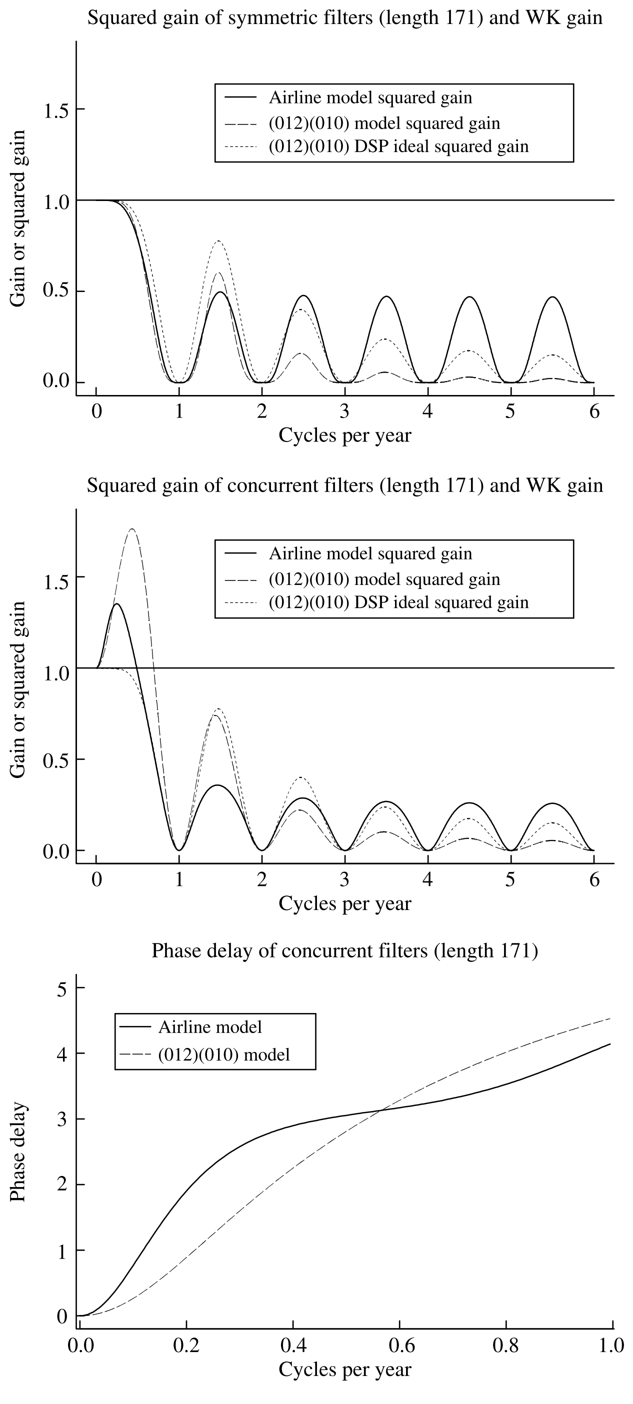

Fig. 10. Squared gains of the symmetric (top) and concurrent filters (middle), and the latter's phase delays (bottom), for the (0, 1, 2) (0, 1, 0) and airline models for the CCIP series of length 171. The squared gain plots also include the DSP ideal, defined as the WK gain function (A.3) of the better-fitting (0, 1, 2) (0, 1, 0) model. For the concurrent filters, comparison with the DSP ideal favors the airline model more often than not. However, the airline model's phase delays are greater over the trend and business cycle frequency range

for this model. However, the phase delay graph of Figure 10 of the concurrent filters shows that this advantage must be weighed against the interpretative disadvantage arising from the airline model's greater phase delay over $0 < \lambda < 0.57$.

A.4. Situations in Which the Mean-Square-Optimal and DSP Approaches Yield Similar Results

For a given λ, $G^{WK}(\lambda)$ and $G^{DSP}(\lambda)$ coincide if and only if $G^{WK}(\lambda)$ has the value zero or one. This occurs for all λ only when the functions $G^{WK}(\lambda)$ and $G^{DSP}(\lambda)$ coincide with the function defined by

$$G^{*}(\lambda) = \begin{cases} 1, & f_N(\lambda) > 0 \\ 0, & f_S(\lambda) > 0 \end{cases} \tag{A.6}$$

Many ideal filters of digital signal processing, e.g., ideal low pass filters, have gain functions with only the values zero and one. The function $G^{*}(\lambda)$ has discontinuous jumps, so a filter with this gain function must have infinitely many coefficients decaying so slowly that their absolute values sum to infinity. Finite filters whose gains approximate $G^{*}(\lambda)$ must be used in practice.

In the AMB seasonal adjustment case, each squared gain function is a rational function of $e^{i\lambda}$ whose maximum value is one. It can take on the values zero and one, and any other values, only finitely many times, not more than the sum of the degrees of the numerator and denominator polynomials. $G^{WK}(\lambda)$ from the canonical decomposition in the sense of Hillmer and Tiao (1982) has the maximal number of λ at which (A.5) holds (among the $G^{WK}(\lambda)$ from all related admissible decompositions) but, being continuous, it can only approximate (A.6) (see Figure 6 for a "close" example produced by $\Theta = 0.95$).

Wecker (1979) derived a formula for the increase in mean squared estimation error when $G^{DSP}(\lambda) \neq G^{WK}(\lambda)$ and the filter with transfer function $G^{DSP}(\lambda)$ is used instead of the WK filter from correct stationary models for Z_t and N_t. Under the assumptions of Bell (1984), this formula can be extended to the ARIMA model case.

A.5. Interpretation of Gains and Phase Delays for Frequency Components

Where they are continuous (see Appendix D), gains and phase delays have simple interpretations for the frequency components of weakly stationary Z_t. From the spectral representation $Z_t = \int_{-6}^{6} e^{i(2\pi/12)\lambda t}\zeta(d\lambda)$ (see Priestley 1981, p. 151), the filter output has the representation

$$\sum_j c_j Z_{t-j} = \int_{-6}^{6} \sum_j c_j e^{i(2\pi/12)\lambda(t-j)}\zeta(d\lambda) = \int_{-6}^{6} e^{i(2\pi/12)\lambda t} C(e^{-i(2\pi/12)\lambda t})\zeta(d\lambda)$$

$$= \int_{-6}^{6} \pm\, G(\lambda) e^{i(2\pi/12)\{\lambda t + \phi(\lambda)\}}\zeta(d\lambda) = \int_{-6}^{6} \pm\, G(\lambda) e^{i(2\pi/12)\lambda\{t-\tau(\lambda)\}}\zeta(d\lambda)$$

(For simplicity, we assume that $\phi(\lambda)$ is differentiable at $\lambda = 0$ so that we can define $\tau(0) = \phi'(0)$. This property can be verified for filters we consider using the decomposition of Appendix D.) Consider some $-6 < \overline{\lambda} < 6$ and $\overline{G} = G(\overline{\lambda}), \overline{\tau} = \tau(\overline{\lambda})$. From continuity at $\overline{\lambda}$, we obtain that for small enough $\varepsilon > 0$,

$$\int_{\overline{\lambda}-\varepsilon}^{\overline{\lambda}+\varepsilon} \pm G(\lambda)e^{i(2\pi/12)\lambda\{t-\tau(\lambda)\}}\zeta(d\lambda) = \pm\overline{G}\int_{\overline{\lambda}-\varepsilon}^{\overline{\lambda}+\varepsilon} e^{i(2\pi/12)\lambda(t-\overline{\tau})}\zeta(d\lambda)$$

$$= \pm\overline{G}B^{\overline{\tau}}\int_{\overline{\lambda}-\varepsilon}^{\overline{\lambda}+\varepsilon} e^{i(2\pi/12)\lambda t}\zeta(d\lambda) \quad \text{(A.7)}$$

This indicates that the filter changes the magnitude of the variate $\int_{\overline{\lambda}-\varepsilon}^{\overline{\lambda}+\varepsilon} e^{i(2\pi/12)\lambda t}\zeta(d\lambda)$, which is effectively the $\overline{\lambda}$-frequency component of Z_t, by the factor $\overline{G}$ approximately and delays the result approximately by $\overline{\tau}$. However, unless restrictions are imposed on $\overline{\lambda}$ and $\overline{\tau}$, there are ambiguities with this interpretation of $\overline{\tau}$, arising from the fact that $e^{-i(2\pi/12)\lambda\overline{\tau}} = e^{-i(2\pi/12)(\lambda\overline{\tau}-12k)}$ for integer k. We only consider the case $\overline{\tau} > 0$, which is the case of interest for concurrent estimates. Suppose the frequency components with $|\overline{\lambda}| < \lambda^*$ are the ones important to the analyst. Then for $\overline{\tau}$ such that $\overline{\tau}\lambda^* < 12$, (A.7) shows unambiguously that the filter, to good approximation (for small enough ε), delays the contribution of the component $\int_{\overline{\lambda}-\varepsilon}^{\overline{\lambda}+\varepsilon} e^{i(2\pi/12)\lambda t}\zeta(d\lambda)$ of Z_t by $\overline{\tau}$. For example, usually $\max_\lambda \tau(\lambda) \le 6$, so this interpretation applies unambiguously to components with frequencies $|\overline{\lambda}| < 2$, which is more than adequate for most analyses. By contrast, for frequency components with $\overline{\lambda} > 12k/\overline{\tau}$ for $k \ge 1$, even when the filter is exactly $\pm\overline{G}B^{\overline{\tau}}$ for some fixed $\overline{G} > 0$ and $\overline{\tau} > 0$, its output $\pm\overline{G}\int_{\overline{\lambda}-\varepsilon}^{\overline{\lambda}+\varepsilon} e^{i(2\pi/12)\lambda(t-\overline{\tau})}\zeta(d\lambda)$ is approximately $\pm\overline{G}\int_{\overline{\lambda}-\varepsilon}^{\overline{\lambda}+\varepsilon} e^{i(2\pi 12)\lambda\{t-(\overline{\tau}-(12k/\overline{\lambda}))\}}\zeta(d\lambda) = \pm\overline{G}B^{\overline{\tau}-(12k/\overline{\lambda})}\int_{\overline{\lambda}-\varepsilon}^{\overline{\lambda}+\varepsilon} e^{i(2\pi/12)\lambda t}\zeta(d\lambda)$, indicating the smaller delay $\overline{\tau} - (12k/\overline{\lambda})$ as an alternative. Where $G(\lambda)$ is nearly zero, the value of $\tau(\lambda)$ is of no interest.

With nonstationary Z_t, we assume that the greater $\tau(\lambda)$ is over an interval where it is unambiguously defined, the more delayed will be the appearance in the filtered series of components with frequencies in the interval, as the experiments with a polygonal turning point of Findley (2000) demonstrate for a suite of trend filters.

Appendix B: Calculating the Filter Coefficients

We obtained the filter coefficients of (1.1) of SEATS and X-11/12-ARIMA by the usual impulse response method (see p. 204 of Bloomfield 2000 or p. 49 of Ladiray and Quenneville 2001), which finds the matrix representation of a linear transformation by applying the transformation to input sequences that are the columns of an identity matrix. For a fixed value of (θ, Θ) and a prescribed length T, SEATS was applied to each of the T impulse series,

$$\delta_t^{(T-u+1)} = \begin{cases} 1, & t = T-u+1 \\ 0, & t \neq T-u+1 \end{cases} \quad 1 \le t \le T \quad \text{(B.1)}$$

specified by $1 \le u \le T$. If we designate the resulting seasonally adjusted series by $N_t^{(u)}$, $1 \le t \le T$, then the coefficients of (1.1) arise as

$$c_{t,-(T-t)+u-1}^T = N_t^{(u)}, \quad 1 \le u \le T \quad \text{(B.2)}$$

The X-11/12-ARIMA seasonal adjustment filter coefficients were obtained analogously, by applying X-11/12-ARIMA, with fixed trend and seasonal filters, to the impulse series extended by twelve backcasts and forecasts from the airline model.

All filter coefficients and model coefficient estimates of this article were obtained from an enhanced version of the X-12-ARIMA program being developed by the U.S. Census Bureau in collaboration with the Bank of Spain which incorporates SEATS (see Monsell, Aston and Koopman 2003). For the case of an airline model with $(\theta, \Theta) = (0.8, 0.8)$, below is an example of a command file (*.spc* file) for X-12-ARIMA/SEATS (a temporary name for the program) for producing a set of output files from which the coefficients of the default X-11/12-ARIMA seasonal adjustment filters, or with a slight change those of the SEATS filters, can be calculated by means of (B.2). The file, which we name *filters88.spc*, will be applied to T input files, each containing one of the series (B.1). The name of each input file must be listed in a *data metafile* whose filename has the extension.*dta*, for example, *Tfilters88.dta*, as described in the X-12-ARIMA documentation. Assuming the X-12-ARIMA/SEATS executable file is named *x12as.exe* and is in the same directory as the data files and data metafile, the command

```
x12as filters88 − d Tfilters88
```

will cause the program to run as specified on all T series of (B.1) and to output each adjusted series into a file having the filename of its input series file but with the extension.*d11* (or.*s11* when the `seats` spec is used). The comment indicator # causes the program to ignore the line with the `seats` spec. If # is instead placed before the `x11` spec, a SEATS adjustment results.

```
series{start = 1970.jan period = 12}
transform{function = none}
arima{model = (0 1 1) (0 1 1) ma = (0.8f 0.8f)}
x11{seasonalma = x11default  trendma = 13  sigmalim = (9.90
9.95) save = (d11)}
#seats{IMEAN = no QMAX = 900 save = s11)}
forecast{maxlead = 12 maxback = 12}
```

Because the series (B.1) are not seasonal and not well modeled by the specified airline model, some default settings have to be overridden. The setting `sigmalim = (9.90 9.95)` in the `x11` spec causes the program to not do extreme value adjustment of the input series $\delta_t^{(T-u+1)}$. For the SEATS adjustments, it is necessary to set IMEAN $= 0$ to prevent mean correction of these input series and also to set QMAX to a high value, e.g., QMAX $= 900$, to make certain that SEATS does not, because of a poor Ljung-Box Q value, reestimate the fixed model coefficients.

Instead of the impulse response function method, to obtain AMB filters the matrix formulas of Bell and Hillmer (1988), which are simplified in McElroy and Sutcliffe (2004), or the state space method of Koopman and Harvey (2003) could be used.

Appendix C: Zeros of Finite-Filter Transfer Functions

Let $Z_t = S_t + N_t, 1 \leq t \leq T$ be a decomposition with ARIMA component models for which the assumptions of Subsection A.1 or the slightly weaker assumptions of Bell and

Hillmer (1988) or McElroy and Sutcliffe (2004) are satisfied. With $\delta_S(B) = 1 - \delta_1^S B - \cdots - \delta_{d_S}^S B^{d_S}$ denoting the differencing polynomial of the ARIMA model for S_t, we now prove that each transfer function of a finite AMB filter estimate $\hat{N}_t$ of N_t contains $\delta_S(e^{-i(2\pi/12)\lambda})$ as a factor. This result, in a slightly different formulation with a less direct proof, is due to William Bell (private communication).

Let F denote the matrix that transforms $[Z_1 \ \cdots \ Z_T]'$ to $[\hat{N}_1 \ \cdots \ \hat{N}_T]'$. Formula (10) of McElroy and Sutcliffe (2004) shows that $F = Q\Delta_S^*$, where Q is a $T \times T - d_s$ matrix and Δ_S^* is the $T - d_S \times T$ band matrix representing the action of $\delta_S(B)$: for $1 \le t \le T - d_S$,

$$\Delta_S^* = \begin{bmatrix} -\delta_{d_S}^S & \cdots & -\delta_1^S & 1 & & \\ & \ddots & \ddots & & \ddots & 1 & \\ & & -\delta_{d_S}^S & \cdots & -\delta_1^S & 1 \end{bmatrix}$$

For $1 \le t \le T$, it follows that the filter for $\hat{N}_t$ of the form (1.1), whose coefficients can be obtained from row t of F, has the decomposition

$$\sum_{j=-(T-t)}^{t-1} c_{j,t}^T B^j = B^{-(T-t)} q_t(B) \delta_S(B) \tag{C.1}$$

with $q_t(z) = \sum_{j=0}^{T-1-d_s} Q_{t,T-d_S-j} z^j$. Hence its transfer function has the factorization $e^{i(2\pi/12)(T-t)\lambda} q_t(e^{-i(2\pi/12)\lambda}) \delta_S(e^{-i(2\pi/12)\lambda})$ with $\delta_S(e^{-i(2\pi/12)\lambda})$ as a factor. Similarly, each AMB signal extraction filter for S_t has a factorization analogous to (C.1) with $\delta_N(B)$ in place of $\delta_S(B)$, hence its transfer function includes $\delta_N(e^{-i(2\pi/12)\lambda})$.

From (A.3), one sees that each zero of $\delta_S(e^{-i(2\pi/12)\lambda})$ of order m is a zero of order $2m$ for the bi-infinite symmetric WK filter transfer function. Numerical results show that most finite symmetric filter transfer functions do not share this property. However, it can be proved for odd-length symmetric filter transfer functions that a zero at $\lambda_0 = 6$ always has order at least two.

Appendix D: Analytical Properties of Concurrent Phase and Phase Delay Functions

We continue with the assumptions and notation of Appendix C, where it was established that an AMB concurrent seasonal adjustment or concurrent trend filter $C(B) = \sum_{j=0}^{T-1} c_j B^j$ has a decomposition

$$C(B) = D(B)\delta_S(B) \tag{D.1}$$

with $D(B) = \sum_{j=0}^{T-1-d_s} d_j B^j$, and we add the assumption that $D(z) \neq 0$ when $|z| \le 1$. We have verified this property for several airline model concurrent seasonal adjustment and trend filters and for the concurrent seasonal adjustment filter of the 1-12-13 model of Section 4 (all with $\delta_S(B) = 1 + B + \cdots + B^{11}$). We also note that from $C(1) > 0$ and $\delta_S(1) > 0$, we obtain $D(1) > 0$. As a result of these properties, $D(B)$ has a phase function $\phi_D(\lambda)$ (such that $D(e^{-i(2\pi/12)\lambda}) = |D(e^{-i(2\pi/12)\lambda})| e^{i(2\pi/12)\phi_D(\lambda)}$) which is continuous on

$-6 \leq \lambda \leq 6$, satisfies

$$\phi_D(-6) = \phi_D(0) = \phi_D(6) = 0 \tag{D.2}$$

and, for $-6 < \lambda < 6$, has a formula relating it to the gain function of $D(B)$,

$$\phi_D(\lambda) = \frac{1}{12} \lim_{\varepsilon \downarrow 0} \left\{ \int_{-6}^{\lambda-\varepsilon} + \int_{\lambda+\varepsilon}^{6} \right\} \log \left| D(e^{-i(2\pi/12)\omega}) \right| \cot \left(\frac{2\pi}{12} \frac{\omega - \lambda}{2} \right) d\omega \tag{D.3}$$

(see Subsection 7.2. of Oppenheim and Schafer 1975).

Using $\phi_{\delta_S}(\lambda)$ to denote a linear phase function of $\delta_S(B)$ (see Section 3.4 of Rabiner and Gold, 1975 for general formulas), we can define a continuous phase function of $C(B)$ via $\phi_C(\lambda) = \phi_D(\lambda) + \phi_{\delta_S}(\lambda)$, leading to $\tau_C(\lambda) = \tau_D(\lambda) + \tau_{\delta_S}(\lambda)$ for the phase delay function and to $\phi_C(6) = \phi_{\delta_S}(6)$ and $\tau_C(6) = \tau_{\delta_S}(6)$ because of (D.2). Thus, the phase properties of $\delta_S(B)$ effectively determine those of $C(B)$ for the highest frequency components. Usually $\delta_S(B) = 1 + B + \cdots + B^{11}$, with

$$\delta_S(e^{-i(2\pi/12)\lambda}) = \left\{ \sin \pi\lambda / \sin \frac{\pi}{12}\lambda \right\} e^{-i(2\pi/12)5.5\lambda} \tag{D.4}$$

for $\lambda \neq 0$, leading to $\phi_{\delta_S}(\lambda) = -5.5\lambda$. This yields the phase delay function $\tau_C(\lambda) = \tau_D(\lambda) + 5.5$, which is graphed in Figure 11 (bottom) along with the squared gains of $\delta_S(B)$ (top) and $D(B)$ (middle) for the concurrent seasonal adjustment and trend filters of the airline model with $(\theta, \Theta) = (0.8, 0.4)$. For the graphs, we calculated $\phi_D(\lambda)$ using the arctangent formula of Section 2 and then $\tau_D(\lambda)$ from $\phi_D(\lambda)$.

The simple filter $\delta_S(B) = 1 + B + \cdots + B^{11}$ strongly suppresses higher frequency components but introduces a large phase delay. Through $D(B)$, the filter $C(B)$ compensates for these features of $\delta_S(B)$. Since $D(B) = \delta_S(B)^{-1}C(B)$, the formula (D.3) reveals that the values of $\phi_C(\lambda)$ and $\tau_C(\lambda)$ at each frequency $\lambda \neq 0, \pm 6$ are influenced by the values of the gain functions of $\delta_S(B)$ and $C(B)$ at all frequencies, not just by their values at frequencies close to λ.

If, instead of using the components of the decomposition (D.1) to define the phase and phase delay of $C(B)$, they are defined directly by the arctangent formula of Section 2, a phase and a phase delay are obtained that have discontinuities at the zeros of $\delta_S(B)$, in particular, at $\lambda = 1$ when $\delta_S(B) = 1 + B + \cdots + B^{11}$. This gives rise to discontinuous phase delay or time shift plots like those of Chapter 8 of Wildi (2004) that are difficult to interpret for $1 < \lambda < 2$, even though, as discussed in Subsection A.5, the continuous phase delay is unambiguously interpretable there.

We shall refer to $D(B)$ as the *minimum delay factor* of the concurrent filter, because our assumption concerning the zeros of $D(z)$ is equivalent to the assumption that $D(B)$ is a minimum delay (or minimum phase lag) filter; see Subsection 7.2 of Oppenheim and Schafer (1975) for several characterizations of such filters and for their formula (7.21) which shows that the continuous phase function $\phi_D(\lambda)$ carries enough information that it determines $|D(e^{-i(2\pi/12)\omega})|$ and therefore $D(e^{-i(2\pi/12)\omega})$ and $D(B)$, up to a positive, constant multiplier. Oppenheim and Schafer also explain that if $D(z)$ has a zero in $|z| < 1$, then $D(B)$ can be written as the product of a minimum delay filter and a filter whose gain function is constant with value one (an *all pass* filter). The latter has a phase delay function

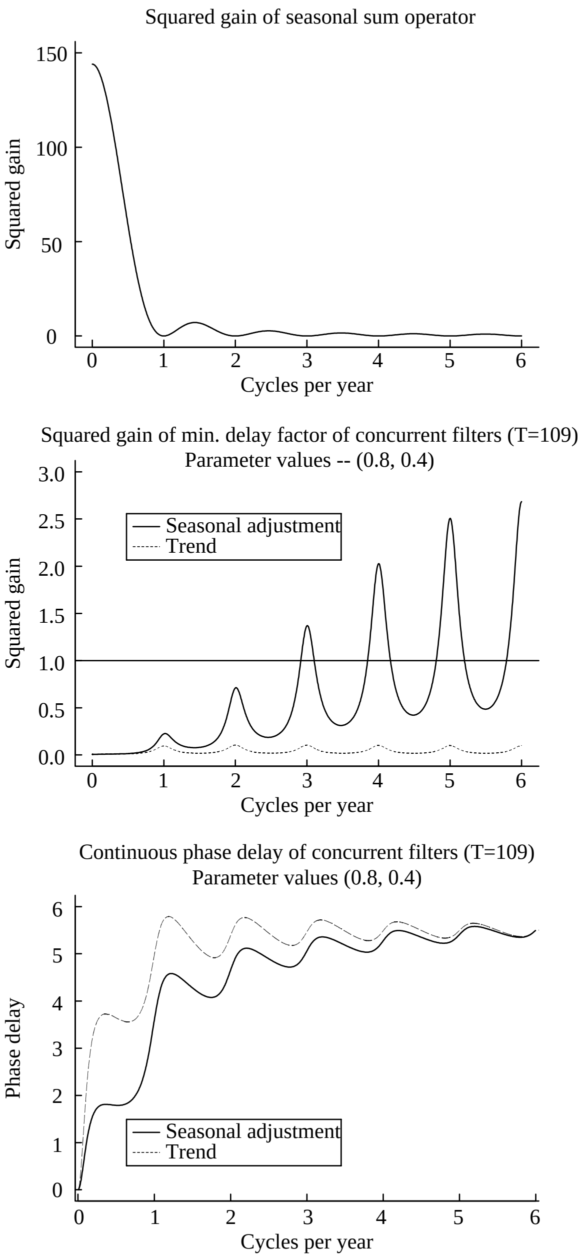

Fig. 11. Diagnostics from the decomposition (D.1) for the length 109 SEATS concurrent seasonal adjustment and trend filters for $(\theta, \Theta) = (0.8, 0.4)$*: squared gains of* $\delta_S(B) = \sum_{j=0}^{11} B^j$ *(top) and of minimum delay factors (middle); continuous phase delays of concurrent filters obtained as the sum of the component phase delays (bottom)*

that is positive at all (nonzero) frequencies. Thus such a $D(B)$ has greater phase delay than its minimum phase factor.

The results cited show that gain and phase properties of concurrent filters are strongly interlinked. Thus it can be impossible to optimize both for interpretability. Section 5.4 and

Chapter 7 of Wildi (2004) describe and apply a concurrent filter design criterion based on minimizing mean square revision error subject to an inequality constraint associated with the phase delay.

7. References

Akaike, H. (1980). Seasonal Adjustment by a Bayesian Modeling. Journal of Time Series Analysis,1, 1–13.

Akaike, H. and Ishiguro, M. (1980). BAYSEA: A Bayesian Seasonal Adjustment Program. Computer Science Monographs No. 13. Tokyo: The Institute of Statistical Mathematics.

Bell, W.R. (1984). Signal Extraction for Nonstationary Time Series. Annals of Statistics, 12, 646–664.

Bell, W.R. (1995). Seasonal Adjustment to Facilitate Forecasting: Arguments for Not Revising Seasonally Adjusted Data. Proceedings of the American Statistical Association, Business and Economic Statistics Section, 268–273.

Bell, W.R. and Hillmer, S.C. (1988). A Matrix Approach to Likelihood Evaluation and Signal Extraction for ARIMA Component Time Series Models. U.S. Bureau of the Census, Statistical Research Division Report Number: Census/SRD/RR-88/22, at www.census.gov/srd/papers/pdf/rr88-22.pdf

Bell, W.R. and Martin, D.E.K. (2004). Computation of Asymmetric Signal Extraction Filters and Mean Squared Error for ARIMA Component Models. Journal of Time Series Analysis, 25, 603–625.

Bell, W.R. and Monsell, B.C. (1992). X-11 Symmetric Linear Filters and Their Transfer Functions. Statistical Research Division Research Report 92/15, U.S. Bureau of the Census, Washington, D.C.

Bloomfield, P. (2000). Fourier Analysis of Time Series: An Introduction. 2nd Ed. New York: Wiley.

Box, G.E.P. and Jenkins, G.M. (1976). Time Series Analysis: Forecasting and Control (revised edition), San Francisco: Holden Day.

Cholette, P.-A. (1979). Spectral Diagnosis and Marginal Improvements of the X-11 Seasonal Adjustment Method for Short Series. Technical Report of the Seasonal Adjustment and Time Series Staff. Ottawa: Statistics Canada.

Dagum, E.B. (1980). The X-11-ARIMA Seasonal Adjustment Method. Catalogue No. 12-564E. Ottawa: Statistics Canada.

Dagum, E.B. (1983). Spectral Properties of the Concurrent and Forecasting Seasonal Filters of the X-11-ARIMA Method. Canadian Journal of Statistics, 11, 73–90.

Dagum, E.B., Chhab, N., and Chiu, K. (1996). Derivation of Properties of the X11ARIMA and Census X11 Linear Filters. Journal of Official Statistics, 12, 329–347.

Dagum, E.B. and Luati, A. (2002). Smoothing Seasonally Adjusted Time Series. Proceedings of the American Statistical Association, Business and Economic Statistics Section [CD-ROM]. Alexandria, VA.

Findley, D.F. (2000). Discussion of Session 14: Trend Estimation. In Proceedings of the International Conference on Establishment Surveys II, 809–811, Alexandria: American Statistical Association.

Findley, D.F., Martin, D.E.K., and Wills, K.C. (2002). Generalizations of the Box-Jenkins Airline Model. Proceedings of the American Statistical Association, Business and Economic Statistics Section [CD-ROM], Alexandria, VA: American Statistical Association.

Findley, D.F. and Martin, D.E.K. (2003). Frequency Domain Analyses of SEATS and X-11/12-ARIMA Seasonal Adjustment Filters for Short and Moderate-length Time Series. Research Report S2003-02, Statistical Research Division, U.S. Census Bureau.

Findley, D.F., Monsell, B.C., Bell, W.R., Otto, M.C., and Chen, B.-C. (1998). New Capabilities and Methods of the X-12-ARIMA Seasonal-adjustment Program. Journal of Business and Economic Statistics, 16, 127–177.

Fischer, B. and Planas, C. (2000). Large Scale Fitting of Regression Models with ARIMA Errors. Journal of Official Statistics, 16, 173–184.

Gómez,V. and Maravall, A. (1994). Program SEATS Signal Extraction in ARIMA Time Series: Instructions for the User. Working Paper ECO 94/28, European University Institute, Florence.

Gómez, V. and Maravall, A. (2000). Seasonal Adjustment and Signal Extraction in Economic Time Series. In A Course in Time Series Analysis, D. Peña, G.C. Tiao, and R.S. Tsay (eds). NY: J. Wiley and Sons.

Harvey, A.C. (1989). Forecasting, Structural Time Series Models, and the Kalman Filter, Cambridge: Cambridge University Press.

Hillmer, S.C. and Tiao, G.C. (1982). An ARIMA model-based Approach to Seasonal Adjustment. Journal of the American Statistical Association, 77, 63–70.

Huot, G., Chiu, K., and Higginson, J. (1986). Analysis of Revisions in the Seasonal Adjustment of Data Using X-11-ARIMA Model-based Filters. International Journal of Forecasting, 2, 217–229.

Kitagawa, G. (1981). A Nonstationary Time Series Model and Its Fitting by a Recursive Filter. Journal of Time Series Analysis, 2, 103–116.

Kitagawa, G. (1985). DECOMP. In TIMSAC-84 Part 2, Computer Science Monographs No. 23. Tokyo: The Institute of Statistical Mathematics.

Koopman, S.J. and Harvey, A.C. (2003). Computing Observation Weights for Signal Extraction and Filtering. Journal of Economic and Dynamic Control, 27, 1317–1331.

Koopman, S.J., Harvey, A.C., Doornik, J.A., and Shephard, N. (1995). Stamp 5.0: Structural Time Series Analyser, Modeller and Predictor, London: Chapman and Hall.

Ladiray, D. and Quenneville, B. (2001). Seasonal Adjustment with the X-11 Method. Lecture Notes in Statistics, no. 158, New York: Springer.

Maravall, A. (1999). Short-term Analysis of Macroeconomic Time Series. In Economics Beyond the Millenium, A.P. Kirman and L.-A. Gérard-Varet, (eds). Oxford: Oxford University Press.

Matas Mir, A. and Rondonotti, V. (2003). The Performance of X-12-ARIMA in the Seasonal Adjustment of Short Time Series. In Seasonal Adjustment. M. Manna and R. Peronnaci, (eds). Frankfurt: European Central Bank.

McElroy, T.S. and Sutcliffe, A. (2004). An Iterated Parametric Approach to Nonstationary Signal Extraction. Statistical Research Division Research Report Series, Statistics #2004–05, U.S. Census Bureau, Washington, D.C. Available at www.census.gov/srd/-papers/pdf/rrs2004-05.pdf

Monsell, B.C., Aston, J.A.D., and Koopman, S.J. (2003). Towards X-13? Proceedings of the American Statistical Association, Business and Economic Statistics Section. [CD-ROM]. Alexandia, VA: American Statistical Association.
Oppenheim, A.V. and Schafer, R.W. (1975). Digital Signal Processing. Engelwood Cliffs: Prentice Hall.
Priestley, M.B. (1981). Spectral Analysis and Time Series. London: Academic Press.
Rabiner, L.R. and Gold, B. (1975). Theory and Application of Digital Signal Processing. Englewood Cliffs: Prentice Hall.
Schwarz, G. (1978). Estimating the Dimension of a Model. Annals of Statistics, 6, 461–464.
Wecker, W.E. (1979). A New Approach to Seasonal Adjustment. Proceedings of the American Statistical Association, Business and Economic Statistics Section, 322–323.
Wildi, M. (2004). Signal Extraction, Efficient Estimation, Unit Roots and Early Detection of Turning Points. Lecture Notes in Economics and Mathematical Systems, Vol. 547. Berlin: Springer.

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