Illuminating Model-Based Seasonal Adjustment with the First Order Seasonal Autoregressive and Airline Models

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Abstract

Stationary first order seasonal autoregressive series are shown to have a canonical model-based decomposition whose estimates have simple formulas from linear regression. The formulas are used to reveal many features of ARMA and ARIMA model-based seasonal adjustment. Our tutorial focus also yields new results, including relative smoothness results based on autocorrelation comparisons of same-calendar-month subseries before and after seasonal adjustment. For a deeper analysis of the SAR(1) decomposition and for generalizations to ARIMA model-based decompositions, the Wiener-Kolmogorov signal extraction filter formulas are developed. These formulas and their ARIMA generalizations by Bell (1984) are applied in several ways. For example, Bell's formulas easily reveal how the seasonal moving average coefficient controls the responsiveness or resistance of ARIMA model-based seasonal adjustments to short term movements in the time series.

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1 Overview

The reader is assumed to be familiar with ARMA and ARIMA models. To **encounter the most fundamental ideas** on a first reading**: Read** this overview through the two paragraphs below (4). Then **peruse** each of the ten sections or subsections whose title has * at the end, also shown in the Contents table above. In each, examine any **Figures** and their captions. Then scan the **formulas** of the section or subsection. Make a guess about which are the most important It is not necessary to follow the details of their derivations. The gist of a derivation is often worth noting. Footnotes can be ignored.

Much of this document revolves around the stationary first order seasonal autoregressive model, or SAR(1). This defines a zero mean process Z_t satisfying

$$Z_t = \Phi Z_{t-q} + a_t, -1 < \Phi < 1, \tag{1}$$

with a zero-mean, uncorrelated (i.e. white noise or w.n.) a_t, whose variance Ea_t^2 is denoted σ_a^2. The autocovariances of Z_t are

$$\gamma_j = EZ_{t+j}Z_t = \sigma_a^2 \begin{cases} \left(1-\Phi^2\right)^{-1}\Phi^k, & |j| = kq, \quad k = 0,1,\ldots \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

See Chapter 9 of Box and Jenkins (1976) for example. Hence the autocorrelations are

$$\rho_j = \begin{cases} \Phi^k, & |j| = qk, \quad k = 0,1,\ldots \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

We only consider $0 < \Phi < 1$ in order to have positive correlation at the seasonal lags $q, 2q, \ldots$. For large enough[1] Φ, (3) shows that Z_t has the fundamental characteristics of a strongly seasonal time series, namely a strong tendency for year-to-year movements in the same direction, with magnitudes (relative to the underlying level, e.g. its mean zero) that evolve mostly slowly. For the monthly case $q = 12$, Figure 1 shows that when $\Phi = 0.95$, then even after 12 years the correlation is greater than 0.5. By contrast, when $\Phi = 0.70$, after five years the correlation is negligible. Such a Z_t would be less recognizably seasonal.

Figure 3 in Subsection 3.2 shows a simulated $\Phi = 0.95$ monthly SAR(1) series Z_t of length 144 with quite seasonal features. It also shows the adjusted series $\hat{N}_t = Z_t - \hat{S}_t$ resulting from removal of the estimate $\hat{S}_t$ of the unobserved signal component S_t of a **signal plus noise decomposition** $Z_t = S_t + N_t$. The signal S_t is specified to have the smallest variance $\gamma_0^S < \gamma_0$ compatible with having the same nonzero-lag autocovariances as Z_t, $\gamma_j^S = \gamma_j, j \neq 0$. This specification will be shown to implicitly specify $N_t = Z_t - S_t$ as white noise with variance $\gamma_0 - \gamma_0^S$. The associated variance reduction $\gamma_0 - \gamma_0^S$ is given by the minimum value of the spectral density (s.d.) of Z_t. This minimum value has a simple formula in the SAR(1) case, as does the s.d., see (9) and (10) in Subsection 3.2.

The graph of $\hat{N}_t$ in Figure 3 appears less smooth than Z_t, and this will be established in a formal way in Subsection 12.2. The signal estimate $\hat{S}_t$ is graphed by calendar month in Figure 4. The $\hat{S}_t$ visibly smooth each of the 12 annual calendar month series of Z_t, a property connected to the fact that the lag 12 autocorrelations of $\hat{S}_t$ are larger than those of Z_t, see Section 12.

For any stationary Z_t with known autocovariances γ_j from an ARMA model for Z_t, the first step toward obtaining linear estimates of a two-unobserved-component decomposition $Z_t = S_t + N_t$ is the determination of an appropriate **autocovariance decomposition**, $\gamma_j = \gamma_j^S + \gamma_j^N, j = 0, 1, \ldots$. The decomposition for the SAR(1) indicated above (with $\gamma_0^N = \gamma_0 - \gamma_0^S$, $\gamma_j^N = 0$, $j \neq 0$) is formally derived in Subsection 3.1. For any finite sample $Z_t, 1 \leq t \leq n$, the autocovariances at lags 0 to $n-1$, furnish a corresponding $n \times n$ covariance matrix decomposition,

$$\begin{aligned} \Sigma_{ZZ} &= \Sigma_{SS} + \Sigma_{NN} \\ [\gamma_{|j-k|}] &= [\gamma_{|j-k|}^S] + [\gamma_{|j-k|}^N]. \end{aligned} \tag{4}$$

This decomposition provides for simplified linear regression formulas (reviewed in Section 4) to yield a decomposition $Z_t = \hat{S}_t + \hat{N}_t$, $t = 1, \ldots, n$ with minimum mean square error linear estimates (estimators)

[1] There is no general agreement on how large Φ should be for an SAR(1) Z_t to be a candidate for seasonal adjustment. This issue is not relevant for our tutorial purposes, so graphical results for component estimates will be presented only for $\Phi = 0.95$, whose Z_t have very visible seasonality.

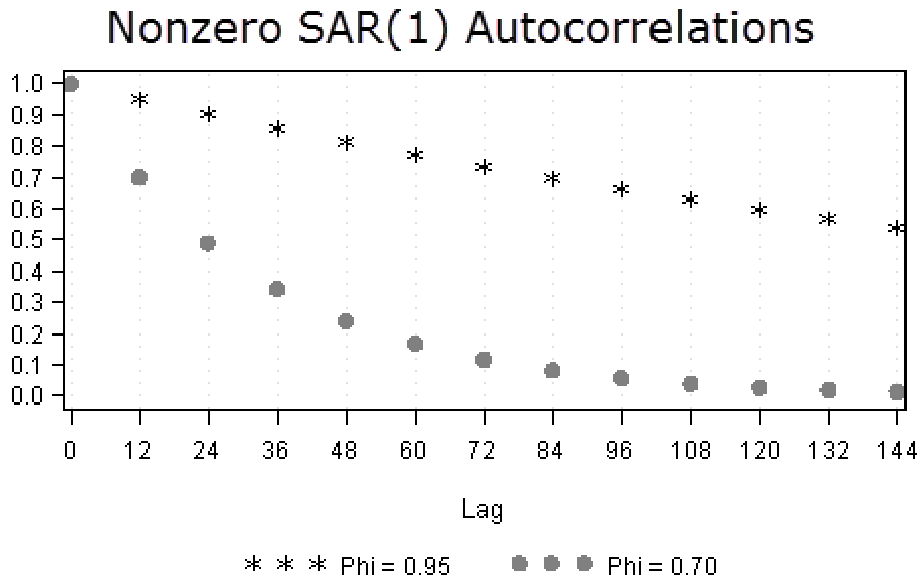

Figure 1: The monthly ($q = 12$) SAR(1)'s nonzero autocorrelations at seasonal lags 12, 24, ..., 144 for two values of Φ. For $\Phi = 0.95$, the autocorrelations are still greater than 0.5 at a lag of twelve years, indicative of well defined and similar seasonal movements for a number of years, as Figure 3 confirms. For $\Phi = 0.70$, they are negligible, after four years, indicating substantially weaker, less consistent "seasonality".

$\hat{S}_t$ and $\hat{N}_t$ of the unobserved components. We refer to such estimates as **MMSE** estimates (Textbooks sometimes call them minimum variance estimates.).Another standard formula provides the variance matrix of their estimation errors.

The two-component SAR(1) decomposition outlined above is important because of the exceptional simplicity of its formulas, derived in Section 5. It can be interpreted as a seasonal plus irregular decomposition of Z_t if the trend/level component of Z_t is taken to be zero, the mean of Z_t. But observe that the graph of Z_t in Figure 3 shows slow up and down movements. This suggests specifying a non-constant level/trend component and estimating a three-component **seasonal, trend, irregular decomposition** for SAR(1) series. In Section 6, SEATS' SAR(1) decomposition is introduced, which is a seasonal-trend-irregular decomposition, $Z_t = s_t + p_t + u_t$. Only the irregular component estimate $\hat{u}_t$ has a simple formula, one which shows that it is a downscaled version of $\hat{N}_t$. The additional algebra required to derive the seasonal and trend components of this decomposition is the starting place for the transition to ARIMA model-based seasonal decompositions.

In Section 7, we introduce the **Wiener-Kolmogorov (W-K) filter formulas** for uncorrelated component estimates from bi-infinite stationary data $Z_t, -\infty < t < \infty$. These immediately reproduce the SAR(1) regression formula estimates of Subsection 5.1 for the intermediate times between the first and last years, $q + 1 \leq t \leq n - q$. Also they easily yield the estimates' ARMA models. Appendix A has quick derivations of the relevant W-K formulas.

In Section 8, after defining the pseudo-spectral density (pseudo-s.d.) of an ARIMA model, we illustrate the kinds of non-stationary W-K calculations done with pseudo-s.d.'s in SEATS and its implementations in other software. This is done by deriving the simple formulas of all of the filters associated with the three-component decomposition of the simplest biannual ($q = 2$) seasonal ARIMA model (82), the model obtained when $\Phi = 1$ in (1), the $(0,1,0)_2$ model. We proceed a little more directly than the tutorial article Maravall and Pierce (1987), which develops fundamental properties of this model's decomposition estimates with somewhat different goals. (In Subsection 12.4, we provide a corrected and extended version of the article's Table 1, which we need for smoothness results.)

In Section 11, W-K formulas are applied to the **Box-Jenkins airline model**. The focus is on the responsiveness or resistance of the seasonal adjustment filters to unusual short term movements in the time series, and how these properties are determined by the seasonal moving average coefficient Θ, much more than by the nonseasonal moving average coefficient θ. Most seasonal ARIMA models contain a seasonal moving average factor $(1-\Theta B^q)$, and the conclusions obtained for the airline model are broadly applicable.

Section 12, the final presentation section, has the most novel material, perhaps more of interest to experienced or technically inclined readers. In an autocorrelation-based way, it shows where **smoothness** is enhanced and where it is reduced among the seasonal decomposition components. Same-calender-month subseries are the main setting. Complete results are presented, first for the two-component SAR(1) decomposition, and then for the irregular component of the nonstationary seasonal $(0,1,0)_2$ model's three-component decomposition. Results are presented for airline model series over an illustrative set of coefficient pairs. For the usual monthly time scale, a few results on the smoothness of trend estimates are described and results on the nonsmoothness of the irregular component are shown.

Bell and Hillmer (1984) provides an excellent historical overview of seasonal adjustment.

2 Some Conventions and Terminology*

A generic primary time series X_t, stationary or not, will be assumed to have $q \geq 2$ observations per year, with the j-th observation for the k-th year having the time index $t = j + (k-1)\,q$, $1 \leq j \leq q$. For simplicity, the series of j-th values from all available years of is called the **j-th calendar month subseries** of X_t even when $q \neq 12$. When $q = 12$, these are the series of January values, the series of February values, etc., 12 series in all. Some seasonal adjustment properties, especially those of seasonal component estimates, are best revealed by the calendar month subseries. When X_t is stationary, the lag k autocorrelation of a calendar month subseries is the lag kq, or k-th seasonal autocorrelation, of X_t. Because some formulas simplify when $q/2$ is an integer, we only consider even q. In our examples $q = 2, 12$. (In practice, $q = 3, 4, 6$ also occur.) Some basic features of canonical ARIMA-model-based seasonal adjustment (**AMBSA** for short) will be related to smoothing the calendar month subseries or detrended versions thereof, see Section 10.2. The definition of **canonical** is given in Subsection 3.2.

Features of **SEATS** referred to are not only features of the model-based seasonal decomposition method and associated auxiliary calculations of TRAMO-SEATS and TSW (Caporello and Maravall, 2004) but are also features of most of the implementations of SEATS in X-13ARIMA-SEATS (U.S. Census Bureau, 2014) and JDemetra+ (Seasonal Adjustment Centre of Competence, 2015).

3 The General Stationary Setting

Seasonal adjustment is an important example of a time series signal extraction procedure. In the simplest setting, the observed series Z_t is treated as the sum of two not directly observable components, the "signal" S_t of interest and an obscuring component, the "noise" N_t,

$$Z_t = S_t + N_t. \tag{5}$$

In the case of stationary Z_t with known autocovariances, γ_j, $j = 0, \pm 1, \ldots$, typically from an ARMA model, estimates of both components can be obtained from an autocovariance decomposition

$$\gamma_j = \gamma_j^S + \gamma_j^N, j = 0, \pm 1, \ldots \ , \tag{6}$$

when γ_j^S and γ_j^N have properties expected of S_t and N_t. Effectively, the additive decomposition (6) implies uncorrelatedness of the signal and noise,

$$ES_{t+j}N_t = 0, j = 0, \pm 1, \ldots, \tag{7}$$

see Findley (2012), which we assume. As a consequence, for a given finite sample $Z_t, 1 \leq t \leq n$, standard linear regression formulas (21) summarized in Section 4 provide a decomposition $Z_t = \hat{S}_t + \hat{N}_t, 1 \leq t \leq n$ with MMSE estimates.

3.1 Autocovariance and Spectral Density Decompositions

The information in an autocovariance sequence γ_j, $j = 0, \pm 1, \ldots$ can be re-expressed, often succinctly and insightfully, by its *spectral density* function (s.d.),

$$g(\lambda) = \sum_{j=-\infty}^{\infty} \gamma_j e^{i2\pi j\lambda} = \gamma_0 + \sum_{j=1}^{\infty} \gamma_j \left(e^{i2\pi j\lambda} + e^{-i2\pi j\lambda}\right) = \gamma_0 + 2\sum_{j=1}^{\infty} \gamma_j \cos 2\pi j\lambda, -1/2 \leq \lambda \leq 1/2.$$

The second and third formulas arise from $\gamma_{-j} = \gamma_j$ and $\cos 2\pi j\lambda = \frac{1}{2}\left(e^{i2\pi j\lambda} + e^{-i2\pi j\lambda}\right)$, the last from Euler's formula $e^{ix} = \cos x + i \sin x$ for real x.

White noise is characterized by having a constant s.d. equal to the variance. An s.d. is nonnegative, $g(\lambda) \geq 0$ always, see (64) for the ARMA formula. It is an even function, $g(-\lambda) = g(\lambda)$, so it is graphed only for $0 \leq \lambda \leq 1/2$. See the SAR(1) example in Figure 2. For any j, the autocovariance γ_j can be recovered from $g(\lambda)$ as

$$\gamma_j = \int_{-1/2}^{1/2} e^{-i2\pi j\lambda} g(\lambda)\, d\lambda = 2\int_{0}^{1/2} \cos 2\pi j\lambda g(\lambda)\, d\lambda, j = 0, \pm 1, \pm 2, \ldots .$$

In particular, the integral of $g(\lambda)$ over $-1/2 \leq \lambda \leq 1/2$ is finite with value γ_0, so the area under a graph of $g(\lambda)$ is $\gamma_0/2$, half the variance (in the units defined by the axes).

An autocovariance decomposition (6) is equivalent to the s.d. decomposition

$$g(\lambda) = g_S(\lambda) + g_N(\lambda), -1/2 \leq \lambda \leq 1/2, \tag{8}$$

with $g_S(\lambda) = \gamma_0^S + 2\sum_{j=1}^{\infty} \gamma_j^S \cos 2\pi j\lambda$ and $g_N(\lambda) = \gamma_0^N + 2\sum_{j=1}^{\infty} \gamma_j^N \cos 2\pi j\lambda$, a key fact.

3.2 The SAR(1) Canonical Signal + Noise Autocovariance Decomposition*

Conceptually attractive and unique decompositions result from the following restriction, introduced by Tiao and Hillmer (1978). An s.d. decomposition with two or more component s.d.'s is called *canonical* if at most one of the components, often a constant (white noise) s.d., has a non-zero minimum. A nonconstant s.d. (or pseudo-s.d. as defined in Section 8) is called canonical if its minimum value is zero. The two-component SAR(1) case provides the simplest seasonal example.

By direct calculation from (2) or from the general ARMA formula (64) below, for a series Z_t with model (1), the s.d. $g(\lambda) = \sigma_a^2 \left(1 - \Phi^2\right)^{-1} \sum_{j=-\infty}^{\infty} \Phi^{|j|} e^{i2\pi jq\lambda}$ has the formula[2]

$$g(\lambda) = \sigma_a^2 \left|1 - \Phi e^{i2\pi q\lambda}\right|^{-2} = \sigma_a^2 \left(1 + \Phi^2 - 2\Phi \cos 2\pi q\lambda\right)^{-1}, -1/2 \leq \lambda \leq 1/2. \tag{9}$$

For $q = 12$, Figure 2 shows an overlay plot of $g(\lambda)$ for the cases $\Phi = 0.70$ and 0.95, each with $\sigma_a^2 = \left(1 - \Phi^2\right)$, which results in $\gamma_0 = 1$ for both SAR(1) processes.

A canonical two-component decomposition of (9) is achieved by separating $g(\lambda)$ from its minimum value,

$$\min_{-1/2 \leq \lambda \leq 1/2} g(\lambda) = \sigma_a^2 (1 + \Phi)^{-2}, \tag{10}$$

which occurs at frequencies in $-1/2 \leq \lambda \leq 1/2$ where $e^{i2\pi q\lambda} = \cos 2\pi q\lambda = -1$, such as $\lambda = \pm (2q)^{-1}$.

[2] Recall that for a complex number $a + ib$, the squared magnitude is $|a + ib|^2 = (a + ib)(a - ib) = a^2 + b^2$.

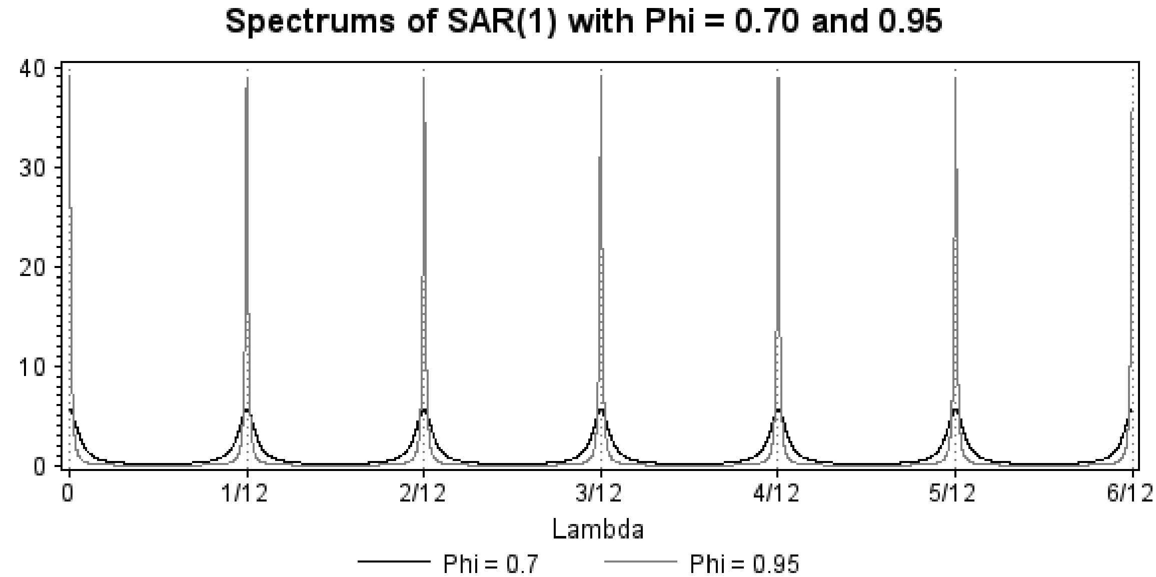

Figure 2: The $q = 12$ SAR(1) spectral densities for $\Phi = 0.95$, and $\Phi = 0.70$ (darker line) with $\sigma_a^2 = \left(1 - \Phi^2\right)$, which results in $\gamma_0 = 1$. So the area under each graph is 1/2 (in the units of the graph). The peaks are at $\lambda = 0$ and at each seasonal frequency, $\lambda = k/12$ cycles per year, $1 \leq k \leq 6$, always with amplitude $\sigma_a^2 (1 - \Phi)^{-2} = (1 + \Phi)(1 - \Phi)^{-1}$. The peaks for $\Phi = 0.70$ are broader and much lower. The miminum value $\sigma_a^2 (1 + \Phi)^{-2} = (1 - \Phi)(1 + \Phi)^{-1}$ occurs midway between each pair of peaks.

The resulting decomposition

$$g(\lambda) = \left(g(\lambda) - \sigma_a^2 (1 + \Phi)^{-2}\right) + \sigma_a^2 (1 + \Phi)^{-2} = g_S(\lambda) + g_N(\lambda) \tag{11}$$

prescribes a matrix decomposition (4) for any sample size $n \geq 1$: With $\Sigma_{ZZ} = \left(1 - \Phi^2\right)^{-1} \sigma_a^2 \left[\Phi^{|j-k|}\right]_{j,k=1,\ldots,n}$ and I the identity matrix of order n,

$$\begin{aligned} \Sigma_{ZZ} &= \left(\Sigma_{ZZ} - \sigma_a^2 (1 + \Phi)^{-2} I\right) + \sigma_a^2 (1 + \Phi)^{-2} I \\ &= \Sigma_{SS} + \Sigma_{NN}, \end{aligned} \tag{12}$$

where Σ_{SS} and Σ_{NN} have the formulas indicated. Substitution into the regression formulas (21) yields estimated signal factors $\hat{S}_t$ and noise factors $\hat{N}_t$ exemplified in Figures 3 and 4 from the simulated SAR(1) $Z_1, \ldots, Z_{144}$ shown. A more informative interpretation of the estimates is developed in Subsections 12.2 and 12.3.

The function $g_S(\lambda) = g(\lambda) - \sigma_a^2 (1 + \Phi)^{-2}$, being nonnegative and even, is the s.d. of a stationary process S_t, an SARMA$(1,1)_q$ it will be shown. We get more insight into the properties of the $\hat{S}_t$ by having a formula for $g_S(\lambda)$ that displays the autocovariances of S_t explicitly:

$$\begin{aligned} g_S(\lambda) &= g(\lambda) - \sigma_a^2 (1 + \Phi)^{-2} = \sigma_a^2 \left\{\left(1 - \Phi^2\right)^{-1} - (1 + \Phi)^{-2}\right\} + 2 \sum_{j=1}^{\infty} \gamma_j \cos 2\pi j \lambda \\ &= \sigma_a^2 \left\{2\Phi \left(1 - \Phi^2\right)^{-1} (1 + \Phi)^{-1}\right\} + 2 \left(1 - \Phi^2\right)^{-1} \sigma_a^2 \sum_{j=1}^{\infty} \Phi^k \cos 2\pi j \lambda. \end{aligned} \tag{13}$$

Thus

$$\gamma_j^S = \Phi\left(1-\Phi^2\right)^{-1}\sigma_a^2 \begin{cases} 2\left(1+\Phi\right)^{-1}, & j=0 \\ \Phi^{k-1}, & |j| = kq, \quad k \geq 1 \\ 0, & |j| \neq 0, kq. \end{cases}, \tag{14}$$

A key feature of S_t with s.d. $g_S(\lambda)$ is that $\gamma_0^S < \gamma_0 = \left(1-\Phi^2\right)^{-1}\sigma_a^2$, because $2\Phi\left(1+\Phi\right)^{-1} < 1$, but $\gamma_j^S = \gamma_j$, $j \neq 0$, so S_t has autocorrelations γ_{kq}^S/γ_0^S proportionately greater than Z_t at all seasonal lags,

$$\rho_j^S = \begin{cases} \frac{1}{2}\left(1+\Phi\right)\Phi^{k-1}, & |j| = kq, \quad k \geq 1 \\ 0, & |j| \neq 0, kq. \end{cases} \tag{15}$$

Such an S_t has the smallest variance compatible with these properties.

The s.d. of N_t is constant,

$$g_N(\lambda) = \sigma_N^2 = \left(1+\Phi\right)^{-2}\sigma_a^2, \tag{16}$$

so the noise component N_t is specified as white noise. Its autocovariances are

$$\gamma_j^N = \begin{cases} \sigma_N^2, & j=0 \\ 0, & |j| > 0, \end{cases} \tag{17}$$

and its order n autocovariance matrix is

$$\Sigma_{NN} = \sigma_N^2 I = \left(1+\Phi\right)^{-2}\sigma_a^2 I. \tag{18}$$

The s.d.'s $g_S(\lambda)$ and $g_N(\lambda)$ from (11) prescribe a signal+noise decomposition of $g(\lambda)$. Because $g_S(\lambda)$ has minimum value zero, S_t is said to be *white noise free*. This decomposition has filter formulas for the MMSE linear estimates of S_t and N_t that are especially simple and revealing[3], as will be seen in Subsection 5.1.

Figure 3 shows the graph of a series Z_t of length 144 simulated from (1) with $q = 12$ and $\Phi = 0.95$, along with its noise component estimates $\hat{N}_t$ from (12) and (21). (All simulations use $\sigma_a^2 = 1$.) The earliest values are assigned the date January, 2002.

Finally, we derive a compact formula for $g_S(\lambda)$ to show a type of calculation that is regularly needed for nonconstant canonical spectral densities. It will be used to identify the component's model.

$$\begin{aligned} g_S(\lambda) &= \left\{\left|1-\Phi e^{i2\pi q\lambda}\right|^{-2} - \left(1+\Phi\right)^{-2}\right\}\sigma_a^2 = \sigma_a^2 \frac{1-\left(1+\Phi\right)^{-2}\left|1-\Phi e^{i2\pi q\lambda}\right|^2}{\left|1-\Phi e^{i2\pi q\lambda}\right|^2} \\ &= \left(1+\Phi\right)^{-2}\sigma_a^2 \frac{\left(1+\Phi\right)^2 - \left|1-\Phi e^{i2\pi q\lambda}\right|^2}{\left|1-\Phi e^{i2\pi q\lambda}\right|^2} \\ &= \Phi\left(1+\Phi\right)^{-2}\sigma_a^2 \frac{\left|1+e^{i2\pi q\lambda}\right|^2}{\left|1-\Phi e^{i2\pi q\lambda}\right|^2}. \end{aligned} \tag{19}$$

It follows from the general formula (64) that S_t has a noninvertible SARMA$(1,1)_q$ model $\left(1-\Phi B^q\right)S_t = \left(1+B^q\right)b_t$ whose white noise b_t has variance $\sigma_b^2 = \Phi\left(1+\Phi\right)^{-2}\sigma_a^2$.

[3]Reasonably simple formulas would result if $\sigma_a^2\left(1+\Phi\right)^{-2}$ in (11) were replaced by a smaller positive number. The canonical decomposition with white-noise-free S_t is not the only possibility. But estimated S_t from such an alternative would be less smooth in the sense of Section 12. (And subtracting a larger number would yield a function with negative values, and thus not an s.d.)

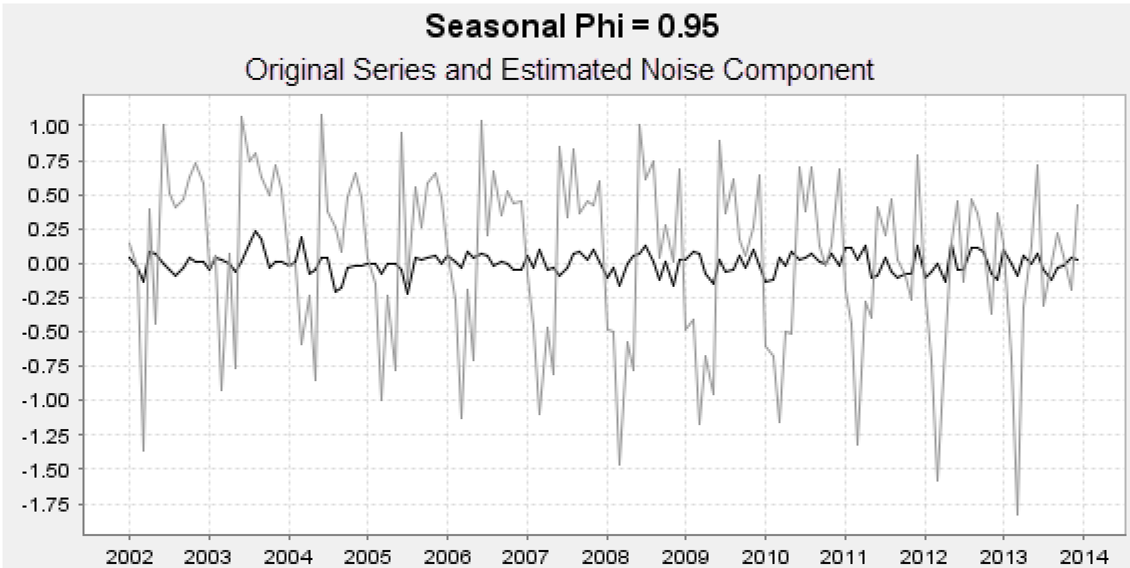

Figure 3: A length 144 simulated monthly $\Phi = 0.95$ SAR(1) series and its estimated noise component $\hat{N}_t$ (darker line) from (21). The series Z_t shows the consistent prominent variations by calendar month seen with quite seasonal time series. The oscillations of $\hat{N}_t$ are considerably smaller yet $\hat{N}_t$ can be considered somewhat less smooth than Z_t after the difference of scale is taken into account, see Subsection 12.2.

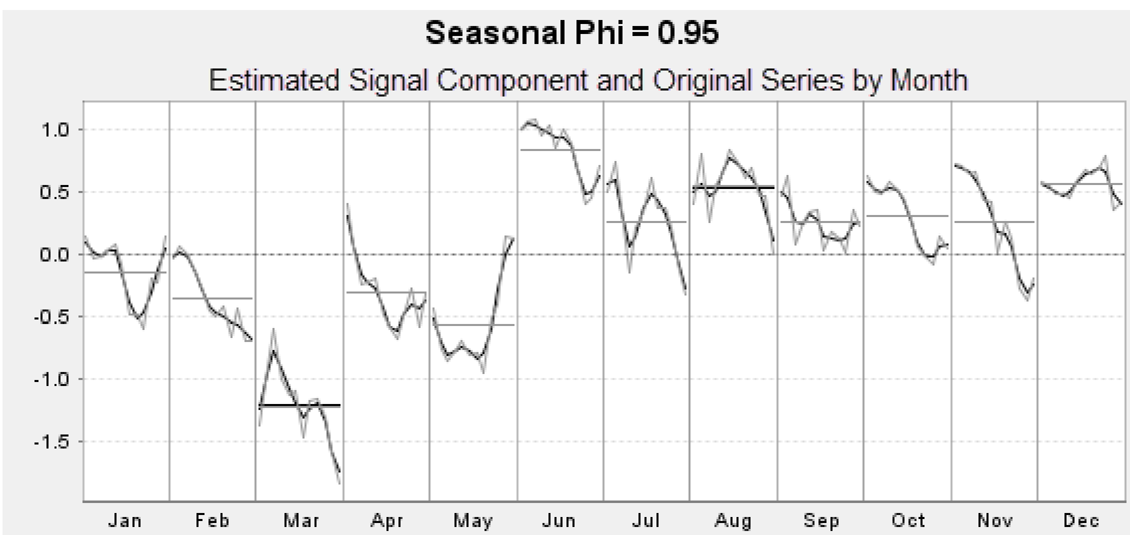

Figure 4: The 12 calendar month subseries of Figure 3 overlaid with their estimated signal component $\hat{S}_t$ values (darker line) from (21). For each month, the horizontal line shows the calendar month average of the $\hat{S}_t$. The $\hat{S}_t$ closely follow all but the most rapid movements of the series, but with fewer changes of direction over the 12 years. Autocorrelation properties help to explain why they evolve somewhat more smoothly than Z_t subseries. Their slightly reduced standard deviations explain why they have slightly reduced extremes, see Subsection 12.3.

4 Regression Formulas for Two-Component Decompositions

Given a column vector of data $Z = (Z_1, \ldots, Z_n)'$, where $'$ denotes transpose, let $S = (S_1, \ldots, S_n)'$ and $N = (N_1, \ldots, N_n)'$ denote the unobserved uncorrelated components of a decomposition $Z = S + N$. From the decomposition of the covariance matrix $\Sigma_{ZZ} = EZZ'$,

$$\Sigma_{ZZ} = \Sigma_{SS} + \Sigma_{NN}, \tag{20}$$

standard linear regression formulas provide MMSE linear estimates $\hat{S}$ of S and $\hat{N}$ of N.

4.1 The Estimated Decomposition*

Because $\Sigma_{SN} = 0$, with 0 denoting the zero matrix of order n, we have $\Sigma_{SZ} = ESZ' = \Sigma_{SS}$. Similarly $\Sigma_{NZ} = \Sigma_{NN}$. Thus the usual regression coefficient formulas $\beta_S = \Sigma_{SZ}\Sigma_{ZZ}^{-1}$ (with $\Sigma_{SZ} = ESZ'$) and $\beta_N = \Sigma_{NZ}\Sigma_{ZZ}^{-1}$ simplify. We have

$$\hat{S} = \beta_S Z, \quad \beta_S = \Sigma_{SS}\Sigma_{ZZ}^{-1}, \quad \hat{N} = \beta_N Z, \, \beta_N = \Sigma_{NN}\Sigma_{ZZ}^{-1}, \quad \beta_S + \beta_N = I. \tag{21}$$

The coefficient formulas result from the fundamental MMSE linear estimation property, the uncorrelatedness of the errors with the data regressor Z,

$$E\left(S - \hat{S}\right) Z' = E\left(N - \hat{N}\right) Z' = 0. \tag{22}$$

The final formula in (21) shows that the estimates yield a decomposition,

$$Z = \hat{S} + \hat{N}. \tag{23}$$

For $1 \leq t \leq n$, the t-th row of β_S provides the filter coefficients for the estimate $\hat{S}_t$ and correspondingly with β_N for $\hat{N}_t$, as will be illustrated in Section 5.

In summary, regression based on (20) provides an observable decomposition of Z in terms of MMSE linear estimates consistent with (20).

4.2 Variance Matrix Formulas

We have $S + N = Z = \hat{S} + \hat{N}$, so if we define $\epsilon = S - \hat{S}$, then

$$N - \hat{N} = -\epsilon, \tag{24}$$

Thus both estimates have the same error covariance matrix,

$$\Sigma_{\epsilon\epsilon} = E\left(S - \hat{S}\right)\left(S - \hat{S}\right)' = E\left(N - \hat{N}\right)\left(N - \hat{N}\right)'. \tag{25}$$

(which ignores any specification/estimation error in the model for Z_t). There are the usual variance decompositions,

$$\Sigma_{SS} = \Sigma_{\hat{S}\hat{S}} + \Sigma_{\epsilon\epsilon}, \quad \Sigma_{NN} = \Sigma_{\hat{N}\hat{N}} + \Sigma_{\epsilon\epsilon}, \tag{26}$$

the first following from the decomposition $S = \hat{S} + \left(S - \hat{S}\right)$, whose components are uncorrelated by (22), and analogously for the second. Following regression terminology, Σ_{SS} can be called *the total variance* of S, $\Sigma_{\hat{S}\hat{S}}$ the *variance of S explained by Z*, and $\Sigma_{\epsilon\epsilon}$ the *residual variance*. Similarly for N with the same residual variance, from (26), which also shows that

$$\Sigma_{SS} - \Sigma_{\hat{S}\hat{S}} = \Sigma_{\epsilon\epsilon} = \Sigma_{NN} - \Sigma_{\hat{N}\hat{N}}, \tag{27}$$

where, from (21),

$$\Sigma_{\hat{S}\hat{S}} = \Sigma_{SS}\Sigma_{ZZ}^{-1}\Sigma_{SS}, \quad \Sigma_{\hat{N}\hat{N}} = \Sigma_{NN}\Sigma_{ZZ}^{-1}\Sigma_{NN}, \tag{28}$$

$$\Sigma_{\epsilon\epsilon} = \Sigma_{\hat{S}\hat{N}} = \Sigma_{SS}\Sigma_{ZZ}^{-1}\Sigma_{NN} = \Sigma_{NN}\Sigma_{ZZ}^{-1}\Sigma_{SS} = \Sigma_{\hat{N}\hat{S}}. \tag{29}$$

The formulas (29) show that the estimates $\hat{S}$ and $\hat{N}$ are positively correlated in the sense that their cross-covariance matrix $\Sigma_{\hat{S}\hat{N}} = \Sigma_{\epsilon\epsilon}$ is positive definite. (As a product of three such matrices, $\Sigma_{\epsilon\epsilon}$ is invertible, hence positive definite.) The estimates are correlated even though S and N are not. In particular $E\hat{S}_t\hat{N}_t > 0$ for each t. (Subsection 12.4 shows that this result does not generalize to the stationary transforms of nonstationary estimates in the ARIMA case.) From (27), the estimates are less variable than their components, precisely due to estimation error.

5 SAR(1) Signal + Noise Decomposition Formulas*

For the SAR(1) model, the entries of the inverse matrix Σ_{ZZ}^{-1} have known, relatively simple formulas, see Wise (1955) and Zinde-Walsh (1988). For example, when $q = 2$, $n = 7$,

$$\begin{aligned}\Sigma_{ZZ}^{-1} &= \sigma_a^{-2}\begin{bmatrix} \frac{1}{1-\Phi^2} & 0 & \frac{\Phi}{1-\Phi^2} & 0 & \frac{\Phi^2}{1-\Phi^2} & 0 & \frac{\Phi^3}{1-\Phi^2} \\ 0 & \frac{1}{1-\Phi^2} & 0 & \frac{\Phi}{1-\Phi^2} & 0 & \frac{\Phi^2}{1-\Phi^2} & 0 \\ \frac{\Phi}{1-\Phi^2} & 0 & \frac{1}{1-\Phi^2} & 0 & \frac{\Phi}{1-\Phi^2} & 0 & \frac{\Phi^2}{1-\Phi^2} \\ 0 & \frac{\Phi}{1-\Phi^2} & 0 & \frac{1}{1-\Phi^2} & 0 & \frac{\Phi}{1-\Phi^2} & 0 \\ \frac{\Phi^2}{1-\Phi^2} & 0 & \frac{\Phi}{1-\Phi^2} & 0 & \frac{1}{1-\Phi^2} & 0 & \frac{\Phi}{1-\Phi^2} \\ 0 & \frac{\Phi^2}{1-\Phi^2} & 0 & \frac{\Phi}{1-\Phi^2} & 0 & \frac{1}{1-\Phi^2} & 0 \\ \frac{\Phi^3}{1-\Phi^2} & 0 & \frac{\Phi^2}{1-\Phi^2} & 0 & \frac{\Phi}{1-\Phi^2} & 0 & \frac{1}{1-\Phi^2} \end{bmatrix}^{-1} \\ &= \sigma_a^{-2}\begin{bmatrix} 1 & 0 & -\Phi & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\Phi & 0 & 0 & 0 \\ -\Phi & 0 & 1+\Phi^2 & 0 & -\Phi & 0 & 0 \\ 0 & -\Phi & 0 & 1+\Phi^2 & 0 & -\Phi & 0 \\ 0 & 0 & -\Phi & 0 & 1+\Phi^2 & 0 & -\Phi \\ 0 & 0 & 0 & -\Phi & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\Phi & 0 & 1 \end{bmatrix}. \end{aligned} \tag{30}$$

For all q and $n \geq q$, as (30) indicates, Σ_{ZZ}^{-1} has a tridiagonal symmetric form, with nonzero values only on the main diagonal and the q-th diagonals above and below. The sub- and superdiagonals have the entries $-\Phi\sigma_a^{-2}$. The first and last q entries of the main diagonal are σ_a^{-2} and the rest are $\sigma_a^{-2}\left(1+\Phi^2\right)$.

For $\beta_N = \sigma_N^2\Sigma_{ZZ}^{-1} = (1+\Phi)^{-2}\sigma_a^2\Sigma_{ZZ}^{-1}$, one has, when $q = 2$, $n = 7$,

$$\beta_N = (1+\Phi)^{-2}\begin{bmatrix} 1 & 0 & -\Phi & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\Phi & 0 & 0 & 0 \\ -\Phi & 0 & 1+\Phi^2 & 0 & -\Phi & 0 & 0 \\ 0 & -\Phi & 0 & 1+\Phi^2 & 0 & -\Phi & 0 \\ 0 & 0 & -\Phi & 0 & 1+\Phi^2 & 0 & -\Phi \\ 0 & 0 & 0 & -\Phi & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -\Phi & 0 & 1 \end{bmatrix}. \tag{31}$$

Further, from $\beta_S = I - \beta_N$,

$$\beta_S = \Phi\left(1+\Phi\right)^{-2}\begin{bmatrix} (2+\Phi) & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & (2+\Phi) & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & (2+\Phi) & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & (2+\Phi) \end{bmatrix}. \tag{32}$$

5.1 The General Filter Formulas

For general q and $n \geq 2q+1$, the Σ_{ZZ}^{-1} formula of Wise (1955) yields the filter formulas for $\hat{N}$ and $\hat{S} = Z - \hat{N}$ shown in (33)–(37) and (38)–(40). For the intermediate times $q+1 \leq t \leq n-q$, the noise component estimate $\hat{N}_t$ is given by a symmetric filter with equal negative initial and final coefficients smaller in magnitude than the positive central coefficient.

$$\hat{N}_t = \frac{1}{\left(1+\Phi\right)^2}\left(-\Phi Z_{t-q} + \left(1+\Phi^2\right) Z_t - \Phi Z_{t+q}\right) \tag{33}$$

The filters for the initial and final years are asymmetric. For the initial year $1 \leq t \leq q$,

$$\begin{aligned} \hat{N}_t &= \frac{1}{\left(1+\Phi\right)^2}\left(Z_t - \Phi Z_{t+q}\right) && (34) \\ &= \frac{1}{\left(1+\Phi\right)^2}\left(-\Phi\left\{\Phi Z_t\right\} + \left(1+\Phi^2\right) Z_t - \Phi Z_{t+q}\right). && (35) \end{aligned}$$

The filter for the final year $n-q+1 \leq t \leq n$ is the time-reverse of the initial year filter,

$$\begin{aligned} \hat{N}_t &= \left(1+\Phi\right)^{-2}\left(-\Phi Z_{t-q} + Z_t\right) && (36) \\ &= \frac{1}{\left(1+\Phi\right)^2}\left(-\Phi Z_{t-q} + \left(1+\Phi^2\right) Z_t - \Phi\left\{\Phi Z_t\right\}\right). && (37) \end{aligned}$$

In comparison with (33) the value $\{\Phi Z_t\}$ in the re-expression (35) appears as the MMSE SAR(1) backcast of the missing Z_{t-q} and, in (37), as the MMSE SAR(1) forecast of the missing Z_{t+q}.

For the signal component estimates $\hat{S}_t$, at intermediate times $q+1 \leq t \leq n-q$ the filter formula is symmetric,

$$\hat{S}_t = \frac{\Phi}{\left(1+\Phi\right)^2}\left(Z_{t-q} + 2Z_t + Z_{t+q}\right) = \frac{4\Phi}{\left(1+\Phi\right)^2}\left(\frac{1}{4}Z_{t-q} + \frac{1}{2}Z_t + \frac{1}{4}Z_{t+q}\right), \tag{38}$$

a downweighted 2×2 seasonal moving average, with weight $4\Phi\left(1+\Phi\right)^{-2}$ tending to 1 when Φ does.

As with $\hat{N}_t$, for the initial and final years, the $\hat{S}_t$ filters are asymmetric[4]. For $1 \leq t \leq q$,

$$\hat{S}_t = \frac{\Phi}{\left(1+\Phi\right)^2}\left(\left(\Phi+2\right) Z_t + Z_{t+q}\right) = \frac{4\Phi}{\left(1+\Phi\right)^2}\left(\frac{1}{4}\left\{\Phi Z_t\right\} + \frac{1}{2}Z_t + \frac{1}{4}Z_{t+q}\right), \tag{39}$$

and for $n-q+1 \leq t \leq n$, the time reverse of the initial year filter,

$$\hat{S}_t = \frac{\Phi}{\left(1+\Phi\right)^2}\left(Z_{t-q} + \left(\Phi+2\right) Z_t\right) = \frac{4\Phi}{\left(1+\Phi\right)^2}\left(\frac{1}{4}Z_{t-q} + \frac{1}{2}Z_t + \frac{1}{4}\left\{\Phi Z_t\right\}\right). \tag{40}$$

[4] An alternative perspective, applicable to any component's estimates from a finite-sample $Z_t, 1 \leq t \leq n$, is that there is just one filter, with coefficients changing over time. Here $\hat{S}_t = \Sigma_{j=1}^{n} c_j\left(t\right) Z_t$, with $c_j\left(t\right)$ defined by (38)–(40). Such time-varying filters form a class of nonlinear filters.

The role of $\{\Phi Z_t\}$ in (39) and (40) is as in (35) and (37). Because the coefficients in (38)–(40) are positive, as are also the autocovariances of Z_t at lags that are multiples of q, it follows that $\hat{S}_t$ and $\hat{S}_{t\pm kq}$ are positively correlated, more strongly than Z_t and $Z_{t\pm kq}$ it will be shown.

5.1.1 Filters Expressed via Backshift Operator Powers and Some Terminology

The coefficient sets in the formulas above all apply for more than one value of t when $n \geq 2q+2$ (Recall that $q \geq 2$.) To reveal this better, let B denote the *backshift operator* defined as follows: for any time series X_t and integer $j \geq 0$, define $B^j X_t = X_{t-j}$ and $B^{-j} X_t = X_{t+j}$ (a *forward shift* if $j \neq 0$) . Since $B^0 X_t = X_t$, one sets $B^0 = 1$. Constant coefficient functions $\Sigma_j c_j B^j$ express (linear, time-invariant) filters. For example, for intermediate-times $q+1 \leq t \leq n-q$, we can rewrite (33) as

$$\hat{N}_t = \frac{1}{(1+\Phi)^2} \left(-\Phi B^q + \left(1+\Phi^2\right) - \Phi B^{-q} \right) Z_t.$$

The intermediate-time filter is now expressed simply as $-\Phi B^q + \left(1+\Phi^2\right) - \Phi B^{-q}$. Its symmetry is established by the fact that for any $j \neq 0$, B^{-j} occurs if and only if B^j does and then both have the same coefficient. It can be applied to Z_t for all t such that $q+1 \leq t \leq n-q$ and to all t when the case of bi-infinite data $Z_\tau, -\infty < \tau < \infty$ is considered.

The one-sided filter that produces the desired estimate for time t without use of later data Z_τ, $\tau > t$, is called the *concurrent filter*. In our finite-sample case, this filter, $(1+\Phi)^{-2}(-\Phi B^q + 1)$, can be applied to Z_t for $q+1 \leq t \leq n$.

5.2 The Error Covariances of the SAR(1) Estimates

For the SAR(1) model, the formulas (27), (18) and (21) yield

$$\begin{aligned} \Sigma_{\epsilon\epsilon} &= \Sigma_{NN} - \Sigma_{\hat{N}\hat{N}} \\ &= \sigma_N^2 \left(I - \sigma_N^2 \Sigma_{ZZ}^{-1} \right) = \sigma_N^2 \left(I - \beta_N \right) = \sigma_N^2 \beta_S. \end{aligned} \tag{41}$$

Hence, for $q = 2$ and $n = 7$, from (18) and (32),

$$\Sigma_{\epsilon\epsilon} = \sigma_a^2 \frac{\Phi}{(1+\Phi)^4} \begin{bmatrix} 2+\Phi & 0 & \Phi & 0 & 0 & 0 & 0 \\ 0 & 2+\Phi & 0 & \Phi & 0 & 0 & 0 \\ \Phi & 0 & 2 & 0 & \Phi & 0 & 0 \\ 0 & \Phi & 0 & 2 & 0 & \Phi & 0 \\ 0 & 0 & \Phi & 0 & 2 & 0 & \Phi \\ 0 & 0 & 0 & \Phi & 0 & 2+\Phi & 0 \\ 0 & 0 & 0 & 0 & \Phi & 0 & 2+\Phi \end{bmatrix}, \tag{42}$$

which reveals the general pattern. The error variances of the initial and final years are larger than the error variance $2\sigma_a^2 (1+\Phi)^{-4} \Phi$ at intermediate times by the amount $\sigma_a^2 \Phi^2 (1+\Phi)^{-4}$. This is the mean square error[5] of using $\Phi (1+\Phi)^{-2} \{\Phi Z_t\}$ to forecast/backcast $\Phi (1+\Phi)^{-2} Z_{t\pm q}$ in (34) and (37), since from (2) we have

$$E\left(Z_{t\pm q} - \Phi Z_t\right)^2 = \left(1+\Phi^2\right)\gamma_0 - 2\Phi\gamma_q^Z = \left(1-\Phi^2\right)\gamma_0 = \sigma_a^2. \tag{43}$$

The intermediate-time mean square error is $\gamma_0^\epsilon = E\left(S_t - \hat{S}_t\right)^2 = E\left(N_t - \hat{N}_t\right)^2 = 2\sigma_N^2 \Phi (1+\Phi)^{-2} = 2\Phi (1+\Phi)^{-4} \sigma_a^2$. On the scale of the variance σ_N^2 of N_t, this is $\gamma_0^\epsilon / \sigma_N^2 = 2\Phi (1+\Phi)^{-2}$, which is approximately

[5] With model-based estimates from more general models for Z_t, more forecasts and backcasts are needed, and their error covariances occur in the mean square error formulas, which are less simple.

0.4997 for $\Phi = 0.95$ and therefore quite substantial, but the ratio decreases to 0 as Φ does. By contrast, for S_t we have $\gamma_0^{\epsilon}/\gamma_0^{S} = (1-\Phi)(1+\Phi)^{-2}$ from (14). This is approximately 0.013 for $\Phi = 0.95$, but this ratio approaches 1.0 as Φ gets small. The fact that the intermediate-time mean square error has the same positive value for all $n \geq 5$ reminds that the error does not become negligible with large n.

6 Three-Component Canonical Decompositions

For seasonal, trend, white noise irregular decompositions,

$$Z_t = s_t + p_t + u_t, \tag{44}$$

a matrix decomposition

$$\Sigma_{ZZ} = \Sigma_{ss} + \Sigma_{pp} + \sigma_u^2 I \tag{45}$$

is needed to obtain the estimated decomposition (46) from (47),

$$Z = \hat{s} + \hat{p} + \hat{u}, \tag{46}$$

$$\hat{s} = \Sigma_{ss}\Sigma_{ZZ}^{-1}Z, \quad \hat{p} = \Sigma_{pp}\Sigma_{ZZ}^{-1}Z, \quad \hat{u} = \sigma_u^2\Sigma_{ZZ}^{-1}Z. \tag{47}$$

For (45), we need a canonical s.d. decomposition (53).

We will use the SAR(1) to illustrate the kind of additional calculations required. A relatively simple exposition is possible only for the biannual case $q = 2$, for which we give the main details. These demonstrate the typical spectral density decomposition calculations done by SEATS. Only the $q = 2$ irregular estimate's filter formulas are as simple as those for the two-component decomposition. A reward for the reader who devotes a bit of attention to the canonical s.d. derivation of the next subsection will come in Section 8, when setting $\Phi = 1$ yields the simple three-component filter formulas of the simplest seasonal ARIMA model.

6.1 SAR(1) with Seasonal Period $q = 2$*

In backshift operator notation, the $q = 2$ model is

$$\left(1 - \Phi B^2\right) Z_t = a_t, \; 0 < \Phi < 1. \tag{48}$$

The autocovariance specifications of SEATS' stationary seasonal and trend components, p_t and s_t, arise from the factorization $1 - \Phi B^2 = \left(1 + \sqrt{\Phi}B\right)\left(1 - \sqrt{\Phi}B\right)$. This factorization leads to the spectral density factorization (53) in which $\left(1 + \sqrt{\Phi}B\right)$ is associated with the seasonal s_t and $\left(1 - \sqrt{\Phi}B\right)$ with the trend p_t. This parallels the ARIMA differencing operator factorization $1 - B^2 = (1+B)(1-B)$, where the role of the year-length sum operator $1 + B$ is to stationarize the seasonal component, and the role of $1 - B$ is to stationarize the trend.

The $q = 2$ version of $g(\lambda)$ in (9) can be correspondingly factored as the product (49) below with $\phi = \sqrt{\Phi}$. The subsequent calculations will be described.

$$g(\lambda) = \frac{\sigma_a^2}{\left|1 - \phi^2 e^{i2\pi 2\lambda}\right|^2} = \frac{\sigma_a^2}{\left|1 + \phi e^{i2\pi\lambda}\right|^2\left|1 - \phi e^{i2\pi\lambda}\right|^2} \tag{49}$$

$$= \frac{1}{2}(1+\phi^2)^{-1}\sigma_a^2\left|1 + \phi e^{i2\pi\lambda}\right|^{-2} + \frac{1}{2}(1+\phi^2)^{-1}\sigma_a^2\left|1 - \phi e^{i2\pi\lambda}\right|^{-2} \tag{50}$$

$$= g_s^*(\lambda) + g_p^*(\lambda) \tag{51}$$

$$= \left\{g_s^*(\lambda) - m^*\right\} + \left\{g_p^*(\lambda) - m^*\right\} + 2m^* \tag{52}$$

$$= g_s(\lambda) + g_p(\lambda) + g_u(\lambda). \tag{53}$$

Expansion of each denominator factor on the right in (49), e.g. $\left|1+\phi e^{i2\pi\lambda}\right|^2 = \left(1+\phi e^{i2\pi\lambda}\right)\left(1+e^{-i2\pi\lambda}\right) = e^{-i2\pi\lambda} + \left(1+\phi^2\right) + e^{i2\pi\lambda}$, followed by partial fraction decomposition (Wikipedia Contributors, 2011), yields that $g(\lambda)$ is the sum (50) of AR(1) spectral densities $g_s^*(\lambda)$ and $g_p^*(\lambda)$ with coefficients $-\phi$ and ϕ respectively and with the same white noise variance $\frac{1}{2}(1+\phi^2)^{-1}\sigma_a^2$ and the same non-zero minimum value,

$$m^* = \kappa(\phi)\,\sigma_a^2, \tag{54}$$

with

$$\kappa(\phi) = \frac{1}{2}(1+\phi)^{-2}(1+\phi^2)^{-1} = \frac{1}{2}\left(1+\sqrt{\Phi}\right)^{-2}(1+\Phi)^{-1}. \tag{55}$$

The reader can directly verify (50). For the canonical decomposition, m^* is subtracted from each s.d. to provide $g_s(\lambda)$ and $g_p(\lambda)$ having minimum value zero as indicated. The remainder $2m^*$ defines

$$g_u(\lambda) = \sigma_u^2 = 2\kappa(\phi)\,\sigma_a^2 \tag{56}$$

in (53), completing $g(\lambda)$'s canonical decomposition.

From (52), for given $Z = (Z_1, \ldots, Z_n)'$, the covariance matrices of the canonical decomposition are

$$\Sigma_{ss} = \Sigma_{ss}^* - m^* I, \quad \Sigma_{pp} = \Sigma_{pp}^* - m^* I, \quad \Sigma_{uu} = \sigma_u^2 I,$$

where Σ_{ss}^* and Σ_{pp}^* are the AR(1) autocovariance matrices determined by $g_s^*(\lambda)$ and $g_p^*(\lambda)$.

For $\hat{u}_t = \sigma_u^2 \Sigma_{ZZ}^{-1}$, at intermediate times, $3 \le t \le n-2$,

$$\hat{u}_t = 2\kappa(\phi)\left\{-\Phi Z_{t-2} + \left(1+\Phi^2\right)Z_t - \Phi Z_{t+2}\right\} = \left(1+\sqrt{\Phi}\right)^{-2}(1+\Phi)^{-1}\hat{N}_t, \tag{57}$$

with ΦZ_t replacing the missing Z_{t-2} and Z_{t+2} at initial and final times, respectively, as in (33)–(37). $\hat{u}_t$ is a downweighting of $\hat{N}_t$ because $\left(1+\sqrt{\Phi}\right)^2 > 1+\Phi$.

The finite-sample formulas for $\hat{s}_t$ and $\hat{p}_t$ in (46) involve left multiplying Σ_{ZZ}^{-1} by covariance matrices of AR(1) processes with coefficients $-\sqrt{\Phi}$ and$\sqrt{\Phi}$, respectively, and thus are algebraically less simple than the two-component formulas. Instead of using (47), we will show a more general approach to deriving their filters in Subsection 7.2.2.

6.2 SAR(1) with q>2

For $q > 2$, with $\phi = \Phi^{1/q}$, the factored form of (9) is

$$g(\lambda) = \sigma_a^2\left|1+\phi e^{i2\pi\lambda} + \cdots + \phi^{q-1}e^{i2\pi(q-1)\lambda}\right|^{-2}\left|1-\phi e^{i2\pi\lambda}\right|^{-2}.$$

The computations parallel those for the $q = 2$ but are more complex and also more typical of computations done by the SEATS with less simple ARMA or ARIMA models. After computing its partial fraction expansion of the form

$$g(\lambda) = \frac{b_0 + \sum_{j=1}^{q-2} b_j\left(e^{i2\pi j\lambda} + e^{-i2\pi j\lambda}\right)}{\left|1+\phi e^{i2\pi\lambda} + \cdots + \phi^{q-1}e^{i2\pi(q-1)\lambda}\right|^2} + \frac{c_0}{\left|1-\phi e^{i2\pi\lambda}\right|^2}, \tag{58}$$

the software subtracts the numerically estimated minimum of each component function in order to obtain $g_s(\lambda)$ and $g_p(\lambda)$. Then it sums the two minima. If the sum is negative the decomposition is *inadmissible*. Otherwise, the *admissible case*, its value defines the white noise s.d. $g_u(\lambda)$ of an s.d. decomposition of the form (52). From this, there are algorithms to compute the entries of the matrices in (45).

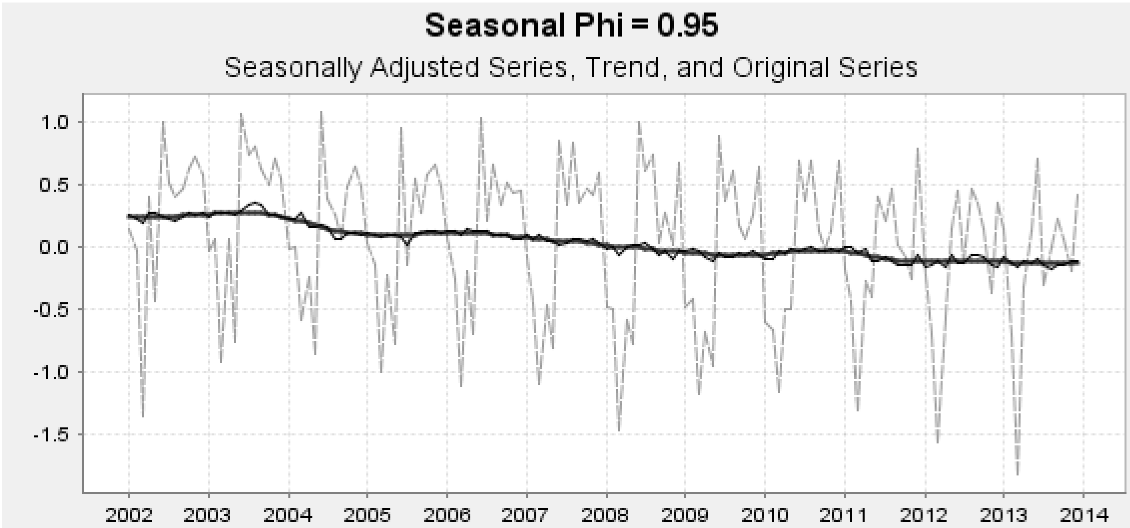

Figure 5: The series of Figure 3, now with the seasonal adjustment $\widehat{sa} = Z - \hat{s}$ (semi-dark line) and trend $\hat{p}$ (darkest line) from its canonical three-component decomposition. $\hat{p}$ can be regarded as having been extracted optimally from $\widehat{sa}$, see Section 10.2.

For the case $q = 12$ and $\Phi = 0.95$, Figure 5 shows the series of Figure 3 together with the canonical seasonal adjustment

$$\widehat{sa} = Z - \hat{s} = \hat{p} + \hat{u} \tag{59}$$

in light bold and the visually smoother canonical trend $\hat{p}$ in thick bold. Section 10.2 establishes that $\hat{p}$ is an optimal estimate of p from $\widehat{sa}$.

7 Wiener-Kolmogorov Formulas and Applications to SAR(1) Decompositions

We return to the two component case (5) with bi-infinite data to show a fundamental and relatively simple approach to obtaining, from a spectral density decomposition, a filter formula for the estimate $\hat{S}_t$ such that for each time t, the error $S_t - \hat{S}_t$ is uncorrelated with $Z_\tau, -\infty < \tau < \infty$. Similarly for $\hat{N}_t = Z_t - \hat{S}_t$. These are the famous Wiener-Kolmogorov signal extraction transfer function formulas.

7.1 Filter Transfer Functions and the Input-Output Spectral Density Formula*

Using the backshift operator B, filter formulas like those above and more general bi-infinite filter expressions $Y_t = \sum_{j=-\infty}^{\infty} \beta_j X_{t-j}$ can be written as $Y_t = \beta(B) X_t$ with filter $\beta(B) = \sum_{j=-\infty}^{\infty} \beta_j B^j$. The s.d. of the filter output series Y_t is related to the input series s.d. $g_X(\lambda)$ by the fundamental formula,

$$g_Y(\lambda) = \left|\beta\left(e^{i2\pi\lambda}\right)\right|^2 g_X(\lambda), \tag{60}$$

see (4.4.3) of Brockwell and Davis (1991). The function $\beta\left(e^{i2\pi\lambda}\right)$ is called the *transfer function* of the filter $\beta(B)$ and $\left|\beta\left(e^{i2\pi\lambda}\right)\right|^2$ is its *squared gain.* When a filter's transfer function $\beta\left(e^{i2\pi\lambda}\right)$ is known, then the filter coefficients can be obtained from it, in general by integration

$$\beta_j = \int_{-1/2}^{1/2} e^{-i2\pi j\lambda} \beta\left(e^{i2\pi\lambda}\right) d\lambda, j = 0, \pm 1, \ldots ,$$

but in practice, for ARMA or ARIMA related transfer functions, by algebraic/numerical algorithms encoded in SEATS.

For example, the transfer function of $\hat{S}_t$ in (38) is

$$\beta_S\left(e^{i2\pi\lambda}\right) = \Phi\left(1+\Phi\right)^{-2}\left(e^{i2\pi q\lambda} + 2 + e^{-i2\pi\lambda}\right) = \Phi\left(1+\Phi\right)^{-2}\left|1+e^{i2\pi q\lambda}\right|^2 .$$

From (60), the spectral density of $\hat{S}_t$ is

$$g_{\hat{S}}\left(\lambda\right) = \Phi\left(1+\Phi\right)^{-2}\left|1+e^{i2\pi q\lambda}\right|^4 g\left(\lambda\right) = \Phi^2\left(1+\Phi\right)^{-4}\sigma_a^2 \frac{\left|1+e^{i2\pi q\lambda}\right|^4}{\left|1-\Phi e^{i2\pi q\lambda}\right|^2}. \tag{61}$$

A stationary ARMA series Z_t has a representation

$$\varphi\left(B\right) Z_t = \vartheta\left(B\right) a_t, \tag{62}$$

with AR and MA polynomials $\varphi\left(B\right) = 1 - \phi_1 B - \cdots - \phi_r B^r$ and $\vartheta\left(B\right) = 1 - \theta_1 B - \cdots - \theta_m B^m$ satisfying[6]

$$\vartheta\left(0\right) = \varphi\left(0\right) = 1, \varphi\left(z\right) \neq 0 \text{ for } \left|z\right| \leq 1, \vartheta\left(z\right) \neq 0 \text{ for } \left|z\right| < 1, \tag{63}$$

where a_t is white noise with variance denoted σ_a^2. (ϑ is script θ.) The general ARMA s.d. formula,

$$g\left(\lambda\right) = \sigma_a^2 \frac{\left|\vartheta\left(e^{i2\pi\lambda}\right)\right|^2}{\left|\varphi\left(e^{i2\pi\lambda}\right)\right|^2}, \tag{64}$$

follows from two applications of (60), see Brockwell and Davis (1991, p.123). Conversely, if (64) and (63) hold, then so does (62) for some white noise process a_t with variance σ_a^2.

This fact can be used to identify ARMA models for bi-infinite data component estimates. For example, from (61) and (64), $\hat{S}_t$ has the noninvertible SARMA$(1,2)_q$ model

$$\left(1 - \Phi B^q\right) \hat{S}_t = \left(1 + B^q\right)^2 b_t, \tag{65}$$

with $\sigma_b^2 = \Phi^2\left(1+\Phi\right)^{-4}\sigma_a^2$. (Note from (64) that an ARMA model is noninvertible, i.e. $\vartheta\left(e^{i2\pi\lambda}\right) = 0$ for some λ, if and only if its spectral density is zero at some λ.)

The next Section shows a versatile way to obtain models and formulas for the bi-infinite data estimates from a decomposition of $g\left(\lambda\right)$.

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[6] The condition on $\vartheta\left(z\right)$ causes no loss of generality in our context and it enables a_t to be identified as the innovations or one-step forecast error process of Z_t given by

$$a_t = Z_t + \sum_{j=1}^{\infty} \pi_j Z_{t-j} = Z_t - \left(-\sum_{j=1}^{\infty} \pi_j Z_{t-j}\right)$$

with $\sum_{j=0}^{\infty} \pi_j z^j = \vartheta\left(z\right)^{-1}\varphi\left(z\right)$ for $\left|z\right| < 1$ $\left(\pi_0 = 1\right)$, see Proposition 4.4.2 for Brockwell and Davis (1991). A nonstandard form of convergence holds in the noninvertible case when $\vartheta\left(z\right)$ has a zero with $\left|z\right| = 1$, see Findley (2012).

7.2 The W-K Formulas*

For a stationary series Z_t with a spectral density decomposition (8) specifying a two-uncorrelated-component decomposition $Z_t = S_t + N_t$, Kolmogorov (1939) and Wiener (1949) independently derived the formulas

$$\beta_S\left(e^{i2\pi\lambda}\right) = \frac{g_S(\lambda)}{g(\lambda)}, \quad \beta_N\left(e^{i2\pi\lambda}\right) = \frac{g_N(\lambda)}{g(\lambda)}, \tag{66}$$

of the transfer functions of each component's MMSE linear estimate

$$\hat{S}_t = \sum_{j=-\infty}^{\infty} \beta_j^S Z_{t-j} = \left(\sum_{j=-\infty}^{\infty} \beta_j^S B^j\right) Z_t, \quad \hat{N}_t = \sum_{j=-\infty}^{\infty} \beta_j^N Z_{t-j} = \left(\sum_{j=-\infty}^{\infty} \beta_j^N B^j\right) Z_t, \tag{67}$$

from bi-infinite data, $Z_\tau, -\infty < \tau < \infty$. A derivation of (66) is provided in Appendix A. For decompositions with more components, the same ratio form applies: each component estimate's transfer function is the ratio of its spectral density to $g(\lambda)$. Ratios of spectral densities are even functions, so such bi-infinite filters are always symmetric, $\beta_{-j}^S = \beta_j^S$, $\beta_{-j}^N = \beta_j^N$.

7.2.1 Re-deriving the SAR(1) Intermediate-time Noise Filter

For the SAR(1), from (9) and (16),

$$\frac{g_N(\lambda)}{g(\lambda)} = \frac{\sigma_N^2}{\sigma_a^2}\left|1 - \Phi e^{i2\pi q\lambda}\right|^2 = (1+\Phi)^{-2}\left\{-\Phi e^{i2\pi q\lambda} + \left(1+\Phi^2\right) - \Phi e^{-i2\pi q\lambda}\right\}. \tag{68}$$

Substituting B^j for $e^{i2\pi j\lambda}$ and B^{-j} for $e^{-i2\pi j\lambda}$ yields

$$\hat{N}_t = (1+\Phi)^{-2}\left\{-\Phi B^q + \left(1+\Phi^2\right) - \Phi B^{-q}\right\} Z_t, \tag{69}$$

the backshift-operator version of intermediate-time filter (33). For such calculations, Maravall and Pierce (1987) introduced a useful convention that permits replacement of $e^{\pm i2\pi q\lambda}$ in transfer functions of the form $\left|\Sigma_j \alpha_j e^{i2\pi q\lambda}\right|^2$ to immediately obtain the backshift and forward-shift operator product formula,

$$\left|\Sigma_j \alpha_j B^j\right|^2 = \left(\Sigma_j \alpha_j B^j\right)\left(\Sigma_j \alpha_j B^{-j}\right). \tag{70}$$

For example, $\left|1-\Phi B^q\right|^2 = \left(1-\Phi B^q\right)\left(1-\Phi B^{-q}\right) = -\Phi B^q + \left(1+\Phi^2\right) - \Phi B^{-q}$, as in (69). We next use (70) to complete the derivation of intermediate time and bi-infinite-data filter formulas of (44) when $q = 2$.

7.2.2 The Estimated Three-Component SAR(1) Decomposition for $q = 2$

This subsection further illustrates the versatility of the W-K formulas as well as the greater algebraic complexity of the trend and seasonal filter formulas of three-component decompositions. The $\hat{u}_t$ formula (57) initially derived using the formula for Σ_{ZZ}^{-1} can now be obtained from the simpler formula (64) by substituting $g_u(\lambda)$ defined by (56) for $g_N(\lambda)$ in (68).

For the estimated trend $\hat{p}_t$, we note from (53) that

$$\begin{aligned} g_p(\lambda) &= \left\{\left|1-\phi e^{i2\pi\lambda}\right|^{-2} - (1+\phi)^{-2}\right\}\sigma^2 = \sigma^2 \frac{1-(1+\phi)^{-2}\left|1-\phi e^{i2\pi\lambda}\right|^2}{\left|1-\phi e^{i2\pi\lambda}\right|^2} \\ &= (1+\phi)^{-2}\sigma^2 \frac{(1+\phi)^2 - \left|1-\phi e^{i2\pi\lambda}\right|^2}{\left|1-\phi e^{i2\pi\lambda}\right|^2} \\ &= \phi\kappa(\phi)\sigma_a^2 \frac{\left|1+e^{i2\pi\lambda}\right|^2}{\left|1-\phi e^{i2\pi\lambda}\right|^2}, \end{aligned} \tag{71}$$

with $\kappa(\phi)$ as in (55). Therefore

$$\frac{g_p(\lambda)}{g(\lambda)} = \kappa(\phi)\left|1+e^{i2\pi\lambda}\right|^2\left|1+\phi e^{i2\pi\lambda}\right|^2 = \kappa(\phi)\left|1+(1+\phi)e^{i2\pi\lambda}+\phi e^{i2\pi 2\lambda}\right|^2.$$

Now (70) yields

$$\begin{aligned}\beta_p(B) &= \kappa(\phi)\left(1+(1+\phi)B+\phi B^2\right)\left(1+(1+\phi)B^{-1}+\phi B^{-2}\right)\\ &= \kappa(\phi)\left\{\phi B^2+(1+\phi)^2 B+2\left(1+\phi+\phi^2\right)+(1+\phi)^2 B^{-1}+\phi B^{-2}\right\}.\end{aligned}$$

Similarly, one can obtain $\beta_s(B)$ by modifying the derivation of $\beta_p(B)$ appropriately. From

$$g_s(\lambda) = \kappa(\phi)\sigma_a^2\frac{\left|1-e^{i2\pi\lambda}\right|^2}{\left|1+\phi e^{i2\pi\lambda}\right|^2}, \tag{72}$$

$$\frac{g_s(\lambda)}{g(\lambda)} = \kappa(\phi)\left|1-e^{i2\pi\lambda}\right|^2\left|1-\phi e^{i2\pi\lambda}\right|^2 = \kappa(\phi)\left|1-(1+\phi)e^{i2\pi\lambda}+\phi e^{i2\pi 2\lambda}\right|^2, \tag{73}$$

one obtains

$$\beta_s(B) = \kappa(\phi)\left\{\phi B^2-(1+\phi)^2 B+2\left(1+\phi+\phi^2\right)-(1+\phi)^2 B^{-1}+\phi B^{-2}\right\},$$

etc. These formulas simplify in the nonstationary case with $\phi = 1$, see Subsection 8.1.2.

7.3 Spectral Density Formulas and Models for the Estimates and Errors

Generalizing the way the $\hat{S}_t$'s model (65) was obtained, we can go from a spectral density decomposition $g(\lambda) = g_S(\lambda) + g_N(\lambda)$ directly to the W-K estimates' spectral densities and models. It follows from (66) and (60) that

$$g_{\hat{S}}(\lambda) = \left(\frac{g_S(\lambda)}{g(\lambda)}\right)^2 g(\lambda) = \frac{g_S^2(\lambda)}{g(\lambda)}, \quad g_{\hat{N}}(\lambda) = \left(\frac{g_N(\lambda)}{g(\lambda)}\right)^2 g(\lambda) = \frac{g_N^2(\lambda)}{g(\lambda)}. \tag{74}$$

Thus, from (74), (68), and (16), the s.d. of $\hat{N}_t$ is

$$g_{\hat{N}}(\lambda) = \frac{\sigma_N^4}{\sigma_a^2}\left|1-\Phi e^{i2\pi q\lambda}\right|^2 = \sigma_a^2(1+\Phi)^{-4}\left|1-\Phi e^{i2\pi q\lambda}\right|^2. \tag{75}$$

So, from (64), $\hat{N}_t$ has an SMA$(1)_q$ model of the form

$$\hat{N}_t = (1-\Phi B^q)c_t, \quad \sigma_c^2 = \sigma_a^2(1+\Phi)^{-4}, \tag{76}$$

where c_t is white noise[7].

[7] In the model formulas for W-K estimates obtained in this way, the model's white noise (innovations) process is different from a_t in (1). At time t it is correlated with future values a_{t+j} for some $j \geq 1$. With (76), where $c_t = (1-\Phi B^q)^{-1}\hat{N}_t$, this is an easily seen consequence of the future value Z_{t+q} in the formula (33) for the estimate at time t , where replacing Z_{t+q} with $\Phi Z_t + a_{t+q}$ and $Z_t - \Phi Z_{t-q}$ with a_t yields $\hat{N}_t = a_t - \Phi a_{t+q} = \left(1-\Phi B^{-q}\right)a_t$, a seasonal MA$(1)_q$ formula involving the future variate a_{t+q}. Thus $c_t = (1-\Phi B^q)^{-1}\left(1-\Phi B^{-q}\right)a_t$. In more detail,

$$c_t = -\Phi a_{t+q} + \left(1-\Phi^2\right)\sum_{j=0}^{\infty}\Phi^j a_{t+2+j}.$$

Forward-in-time models like that for $\hat{N}_t$ play a role in modeling revisions from future data of seasonal adjustment and trend esimates, as demonstrated in Maravall and Pierce (1987).

It is shown in Appendix A that, with decompositions of stationary series, the bi-infinite error processes $\epsilon_t = S_t - \hat{S}_t$ and $-\epsilon_t = N_t - \hat{N}_t$ have the spectral density formula[8]

$$g_\epsilon(\lambda) = \frac{g_S(\lambda) g_N(\lambda)}{g(\lambda)}. \tag{77}$$

This can be used to obtain their shared $\text{SMA}(2)_q$ model $\epsilon_t = \left(1 - (\Phi - 1) B^q - \Phi B^{2q}\right) \nu_t$ with $\sigma_\nu^2 = \Phi (1 + \Phi)^{-4} \sigma_a^2$.

For estimation, it is not necessary to have the models of the components or their estimates or errors, but it is convenient. The models enable the application of standard algorithms to obtain the autocovariances that are needed for estimation, diagnostics, and deeper analyses, such as those related to smoothness and nonsmoothness of estimates presented in Section 12.

Remark on SEATS' Calculation of $\hat{S}_t$ and $\hat{N}_t$. The bi-infinite data filter formulas (67) yielded finite filters for $\hat{S}_t$ and $\hat{N}_t$ above because Z_t has an AR model. A moving average component in (64) would yield a non-constant denominator in (66) and consequently infinite sums, see Subsection 11.1. Such bi-infinite filters might be thought to be impractical, but in SEATS they provide very fast computation of the MMSE seasonal decomposition estimated for finite data[9]. This is made possible by an algorithm of Tunnicliffe-Wilson given in the Appendix of Burman (1980). For any ARMA (or ARIMA) Z_t, using a moderate number of forecasts and backcasts (typically less than 30) it calculates the result of applying the bi-infinite filter with the model's forecasts and backcasts replacing the bi-infinitely many missing past and future values of Z_t.

7.4 Models for the Three-Component SAR(1) Estimates

7.4.1 Case $q = 2$

For the three-component decomposition (44), the model for $\hat{u}_t$ is (76) with the smaller innovation variance of (56), so

$$\rho_j^{\hat{u}} = \rho_j^{\hat{N}} = \begin{cases} -\Phi\left(1 + \Phi^2\right)^{-1}, & j = q \\ 0, & 0 < j \neq q \end{cases}. \tag{78}$$

From (71) and (72),

$$\begin{aligned} g_{\hat{s}}(\lambda) &= \frac{g_s(\lambda)^2}{g(\lambda)} = \Phi \frac{\sigma_u^4}{4\sigma_a^2} \frac{\left|1 - e^{i2\pi\lambda}\right|^4 \left|1 - \phi e^{i2\pi\lambda}\right|^2}{\left|1 + \phi e^{i2\pi\lambda}\right|^2}, \\ g_{\hat{p}}(\lambda) &= \Phi \frac{\sigma_u^4}{4\sigma_a^2} \frac{\left|1 + e^{i2\pi\lambda}\right|^4 \left|1 + \phi e^{i2\pi\lambda}\right|^2}{\left|1 - \phi e^{i2\pi\lambda}\right|^2}. \end{aligned}$$

[8] Note how the formulas for $g_{\hat{S}}(\lambda)$, $g_{\hat{N}}(\lambda)$ and $g_\epsilon(\lambda)$ parallel the finite-sample autocovariance matrix formulas for $\hat{S}_t, \hat{N}_t$ and ϵ_t in (28). For pseudo-spectral density functions of ARIMA models, defined in Subsection 8, the same is true, including the analog of (27). It follows from (64) that W-K estimates $\hat{S}_t$ and $\hat{N}_t$ and their errors $\pm\epsilon_t$ have ARMA models when Z_t does, a result that generalizes to ARIMA decompositions.

[9] However, the error variances in SEATS output usually underestimate the finite-sample error variance because the output variances are theoretical variances from infinite data cases. In the output, the estimates assuming bi-infinite data are called the *historical* or *final estimators*. A *concurrent estimator* in the output is an estimate from data $Z_\tau, \infty < \tau \leq t$. Of special interest is the difference between the concurrent estimator and the final estimator, which is called a *revision estimator*. SEATS calculates its variance and standard error from its ARMA model. For Z_t with the simplest nonstationary seasonal model (82), Section 7 of Maravall and Pierce (1987) illustrates these calculations. See Gómez and Maravall (2001) for additional details and more examples. (Optional output of X-13ARIMA-SEATS provides the finite-sample component error variance matrices, calculated from the regression formulas of McElroy, 2008.)

Hence the models for their estimates are

$$\begin{aligned}
(1+\phi B)\,\hat{s}_t &= (1-B)^2(1-\phi B)\,e_t, \quad \sigma_e^2 = \Phi\frac{\sigma_u^4}{4\sigma_a^2} \\
(1-\phi B)\,\hat{p}_t &= (1+B)^2(1+\phi B)\,\varepsilon_t, \quad \sigma_\varepsilon^2 = \sigma_e^2,
\end{aligned}$$

Not surprisingly, the model for $\hat{s}_t$ differs significantly from the model (65) for $\hat{S}_t$.

The model for $\hat{p}_t$ has an infinite moving average expansion,

$$\begin{aligned}
\hat{p}_t &= (1-\phi B)^{-1}(1+B)^2(1+\phi B)\,\varepsilon_t \\
&= \left(\Sigma_{j=0}^{\infty}\phi^j B^j\right)(1+B)^2(1+\phi B)\,\varepsilon_t = \Sigma_{l=0}^{\infty}\psi_l\varepsilon_{t-l},
\end{aligned}$$

in which $\psi_l > 0$ for all $l \geq 0$. It follows that all autocovariances $\gamma_j^{\hat{p}} = \sigma_\varepsilon^2\sum_{l=0}^{\infty}\psi_l\psi_{l+j}$ and autocorrelations $\rho_j^{\hat{p}}$ of $\hat{p}_t$ are positive, $\rho_j^{\hat{p}} > 0,\ j \geq 0$. This shows that the estimated trend $\hat{p}_t$ is smooth in the sense of Section 12 below.

8 ARIMA Component Filters from Pseudo-Spectral Density Decompositions

For an ARIMA Z_t with differencing operator $\delta(B)$ of degree $d \geq 1$,

$$\varphi(B)\,\delta(B)\,Z_t = \vartheta(B)\,a_t, \tag{79}$$

the *pseudo-spectral density* (pseudo-s.d.) is defined by

$$g(\lambda) = \sigma_a^2\frac{\left|\vartheta\left(e^{i2\pi\lambda}\right)\right|^2}{\left|\delta\left(e^{i2\pi\lambda}\right)\varphi\left(e^{i2\pi\lambda}\right)\right|^2} = \sigma_a^2\frac{\left|\vartheta\left(e^{i2\pi\lambda}\right)\right|^2}{\left|\delta\left(e^{i2\pi\lambda}\right)\right|^2\left|\varphi\left(e^{i2\pi\lambda}\right)\right|^2}. \tag{80}$$

Its integral is infinite because of the λ for which $\delta\left(e^{i2\pi\lambda}\right) = 0$. In the nonstationary signal plus nonstationary noise case, $\delta(B) = \delta_S(B)\,\delta_N(B)$, and $\delta_S\left(e^{i2\pi\lambda}\right)$ and $\delta_N\left(e^{i2\pi\lambda}\right)$ have no common zero. In the seasonal plus nonseasonal case, $\delta_S\left(e^{i2\pi\lambda}\right)$ has zeroes only at seasonal frequencies $\lambda = k/q,\ k = \pm 1, \ldots, q/2$, and $\delta_N\left(e^{i2\pi\lambda}\right) = 0$ only for $\lambda = 0$, as with $\delta_S(B) = 1 + B + \cdots + B^{q-1}$ and $\delta_N(B) = (1-B)^2$ for $\delta(B) = (1-B)(1-B^q)$ of the airline model. The pseudo-s.d. $g(\lambda)$ must be decomposed into sum of seasonal and nonseasonal pseudo-s.d.'s associated with $\delta_S(B)$ and $\delta_N(B)$, respectively.

Under mild assumptions[10], Bell (1984) established the MMSE optimality of the pseudo-spectral generalization of the W-K transfer functions formulas for ARIMA component signal extraction. Tiao and Hillmer (1978), Burman (1980) and Hillmer and Tiao (1982) developed the canonical approach used with extensions and refinements in SEATS and its implementations. The last reference provides a number of examples of canonical seasonal-trend-irregular pseudo-s.d. decompositions[11]

$$g(\lambda) = g_s(\lambda) + g_p(\lambda) + g_u(\lambda). \tag{81}$$

Generalizing the stationary case definition, a pseudo-s.d. decomposition is **canonical** if, with at most one exception, its component pseudo- s.d.'s have minimum value zero.

[10] Bell (1984) also requires that the series $\delta_S(B)\,S_t$ and $\delta_N(B)\,N_t$ be uncorrelated (which can obtained from the more natural s.d. decomposition $g_{\delta(B)Z}(\lambda) = g_{\delta(B)S}(\lambda) + g_{\delta(B)N}(\lambda)$ in practice, see Findley (2012), and Bell's *Assumption A: For $\delta(B)$ of degree d, the initial values, say $Z_1, \ldots, Z_d$, which generate the bi-infinite Z_t from the ARIMA model equation, are uncorrelated with all $\delta(B)\,Z_t$*.This results in correlation (of no practical consequence) between the initial values for the differencing operators of the unobserved S_t and N_t. Formal inference about the initial values is impossible because they are nonstationary and there is one observation of each.

[11] The three-component case, can have stationary u_t that are not white noise but instead an autoregressive "transitory" component, often a cyclical component, see Gómez, V. and A. Maravall (1996) and Kaiser and Maravall (2001). We do not consider these.

8.1 The Simplest Seasonal ARIMA Case*

We derive the canonical decomposition (81) and associated intermediate-time/bi-infinite filters of the simplest SARIMA model, the $(0,1,0)_2$ or lag 2 random walk model,

$$\left(1-B^2\right) Z_t = a_t. \tag{82}$$

Thus $\delta(B) = \left(1-B^2\right) = (1+B)(1-B)$ and the pseudo-s.d. is

$$g(\lambda) = \sigma_a^2 \left|\delta\left(e^{i2\pi\lambda}\right)\right|^{-2} = \sigma_a^2 \left|1+e^{i2\pi\lambda}\right|^{-2} \left|1-e^{i2\pi\lambda}\right|^{-2}. \tag{83}$$

The three-component decomposition of (83) is obtained by setting $\phi = 1$ in (49)–(52), (54) and (56) of Subsection 6.1. The equal minimum values are now $m^* = \sigma_a^2/16$, whence $g_s(\lambda) = g_s^*(\lambda) - m^* = \frac{1}{4}\sigma_a^2 \left\{\left|1+e^{i2\pi\lambda}\right|^{-2} - \frac{1}{4}\right\}$ and $g_p(\lambda) = \frac{1}{4}\sigma_a^2 \left\{\left|1-e^{i2\pi\lambda}\right|^{-2} - \frac{1}{4}\right\}$. It is useful to reexpress s.d.'s as in (80). Here this fundamental calculation takes its simplest form, e.g.

$$\left|1+e^{i2\pi\lambda}\right|^{-2} - \frac{1}{4} = \frac{4-\left|1+e^{i2\pi\lambda}\right|^2}{4\left|1+e^{i2\pi\lambda}\right|^2} = \frac{2-\left\{e^{i2\pi\lambda}+e^{i2\pi\lambda}\right\}}{4\left|1+e^{i2\pi\lambda}\right|^2} = \frac{1}{4}\frac{\left|1-e^{i2\pi\lambda}\right|^2}{\left|1+e^{i2\pi\lambda}\right|^2}.$$

From the analogous calculation for $g_p(\lambda)$, we attain the canonical decomposition (81) with

$$g_s(\lambda) = \frac{\sigma_a^2}{16}\frac{\left|1-e^{i2\pi\lambda}\right|^2}{\left|1+e^{i2\pi\lambda}\right|^2}, \quad g_p(\lambda) = \frac{\sigma_a^2}{16}\frac{\left|1+e^{i2\pi\lambda}\right|^2}{\left|1-e^{i2\pi\lambda}\right|^2}, \quad g_u(\lambda) = \frac{1}{8}\sigma_a^2. \tag{84}$$

This decomposition has already been exposited in Maravall and Pierce (1987) but with some emphases different from ours.

8.1.1 Models, Variances and Autocorrelations of the Components

The formulas (84) reveal the models specified for the components,

$$(1+B)\, s_t = (1-B)\, \tilde{a}_t, \quad (1-B)\, p_t = (1+B)\, \tilde{b}_t, \quad u_t \sim w.n.,\ \sigma_u^2 = \frac{1}{8}\sigma_a^2, \tag{85}$$

with mutually uncorrelated white noise series $\tilde{a}_t$, $\tilde{b}_t$ and u_t having the respective variances $\sigma_a^2/16$, $\sigma_a^2/16$ and $\sigma_a^2/8$ indicated in (84). For later use, we now record the variances (with $\sigma_a^2 = 1$) and lag one autocorrelations of the MA(1) processes
$(1+B)\, s_t = \tilde{a}_t - \tilde{a}_{t-1}$, $(1-B)\, p_t = \tilde{b}_t + \tilde{b}_{t-1}$ and $(1-B)\, sa_t = (1-B)(p_t + u_t) = \tilde{b}_t + \tilde{b}_{t-1} + u_t - u_{t-1}$.

$$\begin{aligned}
\gamma_0^{(1+B)s} &= 2\sigma_{\tilde{a}}^2 = 2\sigma_{\tilde{b}}^2 = \gamma_0^{(1-B)p} = \frac{1}{8}, \quad \rho_1^{(1+B)s} = -\frac{1}{2}, \quad \rho_1^{(1-B)p} = \frac{1}{2}, \\
\gamma_0^{(1-B)sa} &= 2\sigma_{\tilde{b}}^2 + 2\sigma_u^2 = \frac{3}{8}, \quad \gamma_1^{(1-B)sa} = \sigma_{\tilde{b}}^2 - \sigma_u^2 = -\frac{1}{16}, \\
\rho_1^{sa} &= \gamma_1^{(1-B)sa}/\gamma_0^{(1-B)sa} = -\frac{1}{6}.
\end{aligned}$$

8.1.2 W-K Filters for Estimating the Components

Using the W-K formulas as before, e.g. $\beta_s\left(e^{i2\pi\lambda}\right) = g_s(\lambda)/g(\lambda)$, we obtain the filter transfer function formulas

$$\beta_s\left(e^{i2\pi\lambda}\right)=\frac{1}{16}\left|1-e^{i2\pi\lambda}\right|^4,\quad \beta_p\left(e^{i2\pi\lambda}\right)=\frac{1}{16}\left|1+e^{i2\pi\lambda}\right|^4,\quad \beta_u\left(e^{i2\pi\lambda}\right)=\frac{1}{8}\left|1-e^{i2\pi2\lambda}\right|^2. \tag{86}$$

For the seasonal adjustment estimate, we have $\beta_{sa}\left(e^{i2\pi\lambda}\right)=1-\beta_s\left(e^{i2\pi\lambda}\right)=\beta_p\left(e^{i2\pi\lambda}\right)+\beta_u\left(e^{i2\pi\lambda}\right)$. Using (70), we obtain from (86) that the symmetric filters are

$$\beta_s(B)=\frac{1}{16}(1-B)^2\left(1-B^{-1}\right)^2=\frac{1}{16}B^2-\frac{4}{16}B+\frac{6}{16}-\frac{4}{16}B^{-1}+\frac{1}{16}B^{-2}, \tag{87}$$

$$\beta_p(B)=\frac{1}{16}(1+B)^2\left(1+B^{-1}\right)^2=\frac{1}{16}B^2+\frac{4}{16}B+\frac{6}{16}+\frac{4}{16}B^{-1}+\frac{1}{16}B^{-2}, \tag{88}$$

$$\beta_u(B)=\frac{1}{8}\left(1-B^2\right)\left(1-B^{-2}\right)\quad=-\frac{1}{8}B^2+\frac{2}{8}-\frac{1}{8}B^{-2}, \tag{89}$$

$$\beta_{sa}(B)=\beta_p(B)+\beta_u(B)\qquad=-\frac{1}{16}B^2+\frac{4}{16}B+\frac{10}{16}+\frac{4}{16}B^{-1}-\frac{1}{16}B^{-2}. \tag{90}$$

These provide the estimates $\hat{s}_t=\beta_s(B)Z_t$, $\hat{p}_t=\beta_p(B)Z_t$, $\hat{u}_t=\beta_u(B)Z_t$ and $\widehat{sa_t}=\hat{p}_t+\hat{u}_t$ from bi-infinite data, and also for intermediate times $3\le t\le n-2$ from n observations. The preceding formulas, although differently expressed, are equivalent to or are special cases of formulas in Maravall and Pierce (1987). Subsection 12.4 below provides a corrected and expanded version of their Table I. We leave for the interested reader the analogous simpler filter calculations for the two-component decomposition $g(\lambda)=(g(\lambda)-m^{**})+m^{**}=g_S(\lambda)+g_N(\lambda)$, with $m^{**}=\min_\lambda g(\lambda)=1/4$.

Figure 6 shows the similar but more elaborate coefficient pattern of the $q=12$ version of $\beta_{sa}(B)$ for the monthly random walk model $Z_t=Z_{t-12}+a_t$. Also shown is the concurrent filter for estimating sa_t from data Z_τ, $\tau\le t$. Both are very localized, ignoring data more than a year way from the time point of the estimate, and therefore extremely adaptive in a way that is appropriate for series with such erratic trend and seasonal movements.

9 A Inadmissible Seasonal ARIMA Model*

We briefly consider the more general model

$$\left(1-B^2\right)Z_t=\left(1-\Theta B^2\right)a_t \tag{91}$$

in order to show a model whose canonical s.d. decomposition is inadmissible when $\Theta=-0.5$. We take $\sigma_a^2=1$. Partial fraction decomposition of the pseudo-s.d. on the left yields

$$\frac{\left|1+0.5e^{i2\pi2\lambda}\right|^2}{\left|1-e^{i2\pi2\lambda}\right|^2}=\frac{\left|1+0.5e^{i2\pi2\lambda}\right|^2}{\left|1+e^{i2\pi\lambda}\right|^2\left|1-e^{i2\pi\lambda}\right|^2}=\frac{7/16}{\left|1+e^{i2\pi\lambda}\right|^2}+\frac{7/16}{\left|1-e^{i2\pi\lambda}\right|^2}-\frac{1}{4},$$

not a pseudo-s.d. decomposition because $-1/4$ is not an s.d. Since $\min_\lambda\left|1+e^{i2\pi\lambda}\right|^{-2}=\min_\lambda\left|1-e^{i2\pi\lambda}\right|^2=1/4$, the canonical decomposition is

$$\frac{\left|1+0.5e^{i2\pi2\lambda}\right|^2}{\left|1-e^{i2\pi2\lambda}\right|^2}=\frac{7}{16}\left\{\frac{1}{\left|1+e^{i2\pi\lambda}\right|^2}-\frac{1}{4}\right\}+\frac{7}{16}\left\{\frac{1}{\left|1+e^{i2\pi\lambda}\right|^2}-\frac{1}{4}\right\}+\left\{\frac{14}{64}-\frac{16}{64}\right\}.$$

It is inadmissible because the final bracketed term has the negative value $-2/64$, not a possible white noise variance. Hillmer and Tiao (1982) show (91) has an admissible decomposition for $\Theta\ge-.1716$.

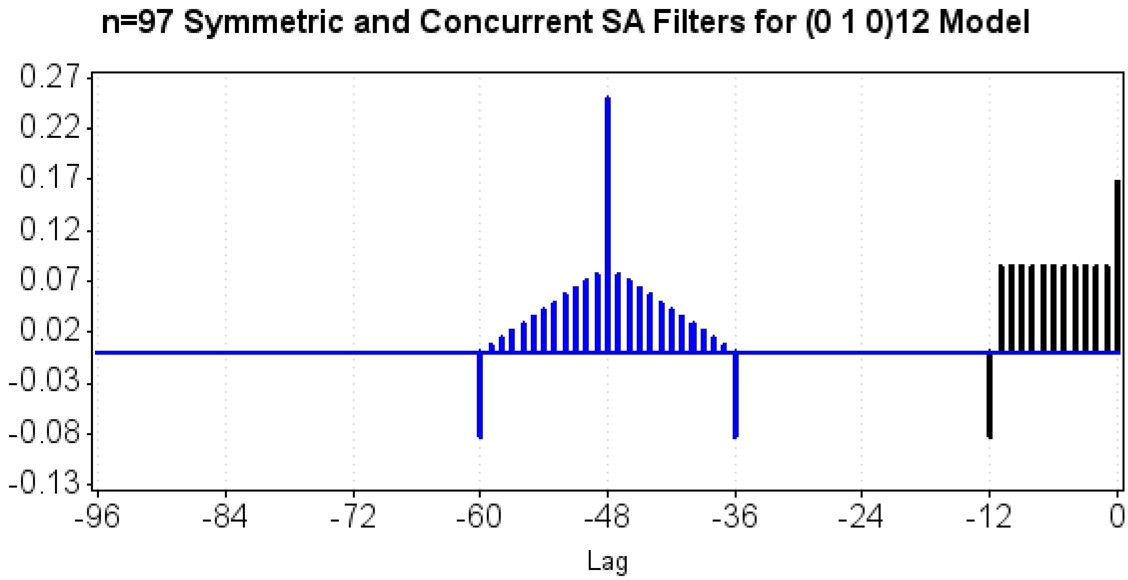

Figure 6: Concurrent and symmetric $n = 97$ seasonal adjustment filters of the monthly seasonal random walk, the $(0,1,0)_{12}$ model. The coefficients of this $q = 12$ symmetric filter, shown for the midpoint $t = 48$ of the interval $0 \leq \tau \leq 96$, are, as with the $q = 2$ formula (90), positive at time t and decreasing to the negative coefficients at times $t \pm q$. The filters for intermediate times $12 \leq t \leq 84$ have the same coefficients. The symmetric filter and the concurrent filter respond only to data within a year of t, and so are very adaptive. The coefficient sign change for the same calendar month a year away from t can result in large revisions.

10 Some General ARIMA Canonical Decomposition Results

10.1 What Seasonal Decomposition Filters Remove or Preserve*

Observe that, among the filters (87)–(90), the coefficients of the seasonal and irregular filters $\beta_s(B)$ and $\beta_u(B)$ sum to zero. Therefore these filters will annihilate a constant level component, e.g. the sample mean. By contrast, the coefficients of the trend and seasonal adjustment filters $\beta_p(B)$ and $\beta_{sa}(B)$ sum to one. They will reproduce a constant level component. These are completely general results, applying to AMBSA filters, finite or infinite, symmetric or asymmetric, from any ARIMA model whose differencing operator has $1 - B^q = (1 - B)U(B)$, $q \geq 2$, with $U(B) = 1 + B + \cdots B^{q-1}$ as a factor. The tables of Bell (2012) cover also several generations of symmetric and asymmetric X-11 filters. In many cases, linear functions, and in exceptional cases covered in Bell (2015), even higher degree polynomials in t are eliminated by $\beta_s(B)$ and $\beta_u(B)$ and preserved by $\beta_p(B)$ and $\beta_{sa}(B)$.

Here are some illustrative calculations, also for $q = 2$ fixed seasonal effects $\alpha(-1)^t$. From (87) and $\left(1 - B^{-1}\right) = B^{-1}(B - 1)$ we obtain that $\beta_s(B) = B^{-2}(1 - B)^4$. Differencing lowers the degree of a polynomial by one, e.g., $(1 - B)t^3 = t^3 - (t-1)^3 = 3t^2 - 3t + 1$. Hence $\beta_s(B)$ will annihilate a cubic component αt^3, whereas $\beta_{sa}(B) = 1 - \beta_s(B)$ will reproduce it. The filter $\beta_s(B)$ will reproduce a stable seasonal component $\alpha(-1)^t$,

$$\left\{\frac{1}{16}B^2 - \frac{4}{16}B + \frac{6}{16} - \frac{4}{16}B^{-1} + \frac{1}{16}B^{-2}\right\}(-1)^t = (-1)^t\left\{\frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} + \frac{1}{16}\right\} = (-1)^t,$$

because $B^2(-1)^t = (-1)^{t-2} = (-1)^t$, whereas $B(-1)^t = (-1)^{t-1} = -(-1)^t$. Correspondingly, $\beta_{sa}(B)$ will annihilate $\alpha(-1)^t$. The reader can verify that $\beta_p(B)\alpha(-1)^t = 0$ and that the irregular filter $\beta_u(B) = -\frac{1}{8}B^{-2}(1+B)^2(1-B)^2$ will eliminate both a linear trend and a stable seasonal component.

10.2 Reciprocal Smoothing Properties of Seasonal and Trend Estimates

Consider the detrended series from (46) and (47),

$$Z - \hat{p} = \hat{s} + \hat{u}. \tag{92}$$

The matrix formulas (47) easily yield $\hat{s} = \Sigma_{ss}(\Sigma_{ss} + \Sigma_{uu})^{-1}(Z - \hat{p})$. It follows that $\hat{s}$, in addition to being the MMSE linear estimate of s from Z, is also the MMSE linear function of $Z - \hat{p}$ for estimating s from $Z - \hat{p}$, because its error $s - \hat{s}$, being uncorrelated with Z, is uncorrelated with $Z - \hat{p}$, a linear function of Z. Analogous reasoning yields that the estimated trend $\hat{p}$ is an MMSE linear estimate of p from the seasonally adjusted series $Z - \hat{s}$, as asserted in the discussion preceding Figure 5.

These stationary case results are special cases of the results of Theorem 1 and Remark 4 of McElroy and Sutcliffe (2006) for ARIMA Z_t. Further, under correct model assumptions, their paper also provides convergence results to the MMSE estimates of trend and seasonal for iterations starting with a non-MMSE estimate of trend (or seasonal).

For the canonical seasonal-trend-irregular decomposition of a series simulated from the Box-Jenkins airline model[12],

$$(1-B)(1-B^q)Z_t = (1-\theta B)(1-\Theta B^q)a_t = a_t - \theta a_{t-1} - \Theta a_{t-q} + \theta\Theta a_{t-q-1}, \tag{93}$$

with $\theta = \Theta = 0.60$, Figure 7 shows how the $\hat{s}_t$ visually smooth the detrended calendar month subseries. The perspective that trend should be estimated from a deseasonalized series and the seasonal component should be estimated from a detrended series is central to the iterative X-11 method for estimating these components, see Ladiray and Quenneville (2001).

10.3 Two-Component Decompositions

When the trend is not of interest, the focus is the two-component seasonal-nonseasonal decomposition, $Z_t = s_t + n_t$, with n_t the non-seasonal component, whose estimate is the seasonal adjustment, $\hat{n}_t = \widehat{sa}_t = Z_t - \hat{s}_t$. With some nonseasonal Z_t, trend plus cycle decompositions are of interest. Maravall and Planas (1999) derive a variety of interesting properties of the canonical decomposition in such cases. For example, in the bi-infinite case, the canonical decomposition estimates have the smallest error variances.

The results of Bell (1984) show for the ARIMA component W-K estimation case that error processes are stationary. For example, applied to the seasonal s_t and nonseasonal n_t case with $\delta(B) = \delta_s(B)\delta_n(B)$ where $\delta_s(B) = U(B) = 1 + B + \cdots B^{q-1}$ and $\delta_n(B) = (1-B)^2$, the error processes $s_t - \hat{s}_t$ and $n_t - \hat{n}_t$ are stationary and have the spectral density of the estimation error process of the stationary decomposition

$$\delta(B)Z_t = \delta(B)\hat{s}_t + \delta(B)\hat{n}_t = (1-B)^2\{\delta_n(B)\hat{s}_t\} + U(B)\{\delta_n(B)\hat{n}\}.$$

That is,

$$g_\epsilon(\lambda) = \frac{g_{\delta_n(B)s}(\lambda)g_{\delta(B)n}(\lambda)}{g_{\delta(B)Z}(\lambda)}.$$

It follows from Appendix A that $\delta(B)\hat{s}_t$ and $\delta(B)\hat{n}_t$ are always positively correlated. This is not true of the minimally differenced stationary transforms, $\delta_s(B)\hat{s}_t$ and $\delta_n(B)\hat{n}_t$, see Subsection 12.4.

[12] The airline model is the most commonly chosen model by TRAMO's automatic model idenfication procedure (which is implemented somewhat differently in different SEATS implementations). This model was developed by Box and Jenkins for obviously seasonal series like the January, 1949 to December, 1960 International Airline Passenger series shown in the background of Figure 16. Its values are given in Part V of Box and Jenkins (1976).

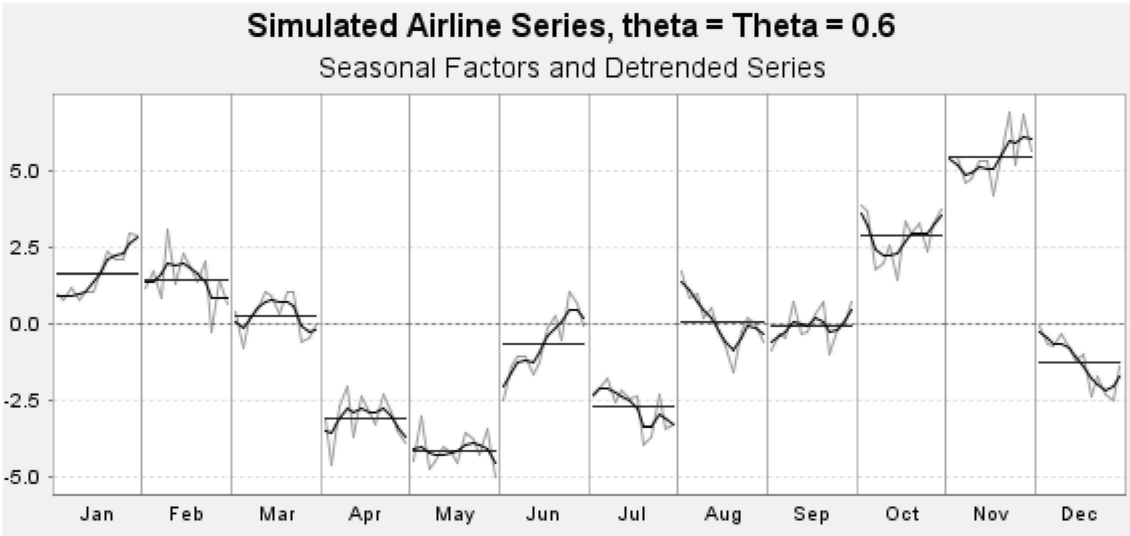

Figure 7: The seasonal factors $\hat{s}_t$ (darker lines) are the MMSE estimates of the detrended series $Z_t - \hat{p}_t$ (lighter lines) from the simulated airline series Z_t with $\theta = \Theta = 0.6$. The horizontal lines are the calendar month means of the $\hat{s}_t$.

11 Airline Model Results

We continue with (93) and special cases thereof to demonstrate important aspects of AMBSA. Hillmer and Tiao (1982) show that, when $\Theta \geq 0$, the airline model is admissible for all $-1 \leq \theta \leq 1$. There are admissible decompositions for some negative $\Theta \geq -0.3$ for a Θ-dependent interval of θ values. We only consider $\Theta \geq 0$.

11.1 How Adaptive or Resistant are Estimates to Data Changes: Effect of Θ

In practice, ARIMA models that are identified for seasonal time series usually have a seasonal moving average factor $1 - \Theta B^q$, and the influence of Θ conforms to the conclusions we obtain in this Subsection. When the model for Z_t, whether ARMA or ARIMA, has such a moving average factor, then the W-K filters are bi-infinite rather than finite, as will be seen. Their coefficients at seasonal lags decay slowly if Θ is large, a property that is replicated in the finite-sample filters. Then the seasonal adjustment adapts slowly to rapid changes and resists the influence of changes that are not long lasting. The airline model filter coefficient figures of this section illustrate this clearly, and also show that Θ has a much more dominant role than the nonseasonal parameter θ in determining how adaptive or resistant the seasonal adjustment estimates are to sudden changes in the time series. We start with an analytical example to make clear why this is so.

The transfer function of the irregular component filter for the monthly ($q = 12$) airline model,

$$\beta_u\left(e^{i2\pi\lambda}\right) = \frac{\sigma_u^2}{g(\lambda)} = \frac{\sigma_u^2 \left|\delta\left(e^{i2\pi\lambda}\right)\right|^2}{\left|1 - \theta e^{i2\pi\lambda}\right|^2 \left|1 - \Theta e^{i2\pi 12\lambda}\right|^2},$$

leads to the filter $\beta_u(B)$ with the factored form

$$\beta_u(B) = \sigma_u^2 \left\{\delta(B)\,\delta\left(B^{-1}\right)\right\} \left\{(1 - \theta B)^{-1}\left(1 - \theta B^{-1}\right)^{-1}\right\} \left\{\left(1 - \Theta B^{12}\right)^{-1}\left(1 - \Theta B^{-12}\right)^{-1}\right\}. \tag{94}$$

From $\delta(B) = (1 - B)\left(1 - B^{12}\right)$, the first bracketed factor defines a finite symmetric filter (of length 27), which contains $\delta(B)$, ensuring that $\hat{u}_t = \beta_u(B) Z_t$ will be a stationary time series. The second and third factors make $\beta_u(B)$ bi-infinite. For example,

$$\left(1-\Theta B^{12}\right)^{-1}\left(1-\Theta B^{-12}\right)^{-1} = \left(\Sigma_{j=0}^{\infty}\Theta^{j}B^{12j}\right)\left(\Sigma_{j=0}^{\infty}\Theta^{j}B^{-12j}\right)$$

$$= \Sigma_{j=0}^{\infty}\Theta^{2j}\left(1+\Sigma_{j=1}^{\infty}\Theta^{j}\left(B^{12j}+B^{-12j}\right)\right) = \left(1-\Theta^{2}\right)^{-1}\Sigma_{j=-\infty}^{\infty}\Theta^{|j|}B^{12j}. \quad (95)$$

Similarly,

$$(1-\theta B)^{-1}\left(1-\theta B^{-1}\right)^{-1} = \left(1-\theta^{2}\right)^{-1}\Sigma_{j=-\infty}^{\infty}\theta^{|j|}B^{j}.$$

Thus the effect of Θ on the symmetric filter coefficients is largest at seasonal leads and lags $k = \mp 12j$, with the effect decaying at the rate $\Theta^{|j|/12}$. The effect of θ decays at the much faster rate $\theta^{|j|}$. The nonzero values in Figure 1 illustrate the decay rate of $\Theta^{k}, k = 0, \ldots, 12$ for $\Theta = 0.70$ and 0.95.

This decay rate difference leads to a frequently observed feature of seasonal adjustment filters that will be displayed in graphs of airline model filters and associated seasonal factors. For the estimate at time t, the observation Z_t gets the largest coefficient and the next largest coefficients are usually for the nearest same-calendar-month values $Z_{t\pm 12k}$ in the observation interval $1 \le t \le n$, with coefficient magnitudes decreasing exponentially with k. The greater the value of Θ is, the less localized and adaptive but more stable the estimate will be. When Θ is small, the filters are quite localized and adaptive but more likely to have large revisions when data at time $n+12$, $n+24$, etc. are first adjusted.

11.2 Seasonal Filters from Various θ,Θ and Their Factors for the International Airline Passenger Data*

For a fixed finite data size n, the dominating effect of Θ is more clearly observed with finite-sample asymmetric filters, especially with the one-sided concurrent seasonal adjustment filters for the most recent time $t = n$. Figures 8–11 show the concurrent seasonal adjustment filter coefficients for $n = 97$ from airline models with $(\theta, \Theta) = (0.6, 0.0)$, $(0.6, 0.3)$, $(0.9, 0.6)$, $(0.9, 0.9)$. In all cases, same-calendar-month values Z_{n-12k}, $k = 0, 1, \ldots$ receive much larger weight than values from other months, with greater positive weight for the current month n for larger[13] Θ . Seasonal factor and seasonal adjustment filters are related by $\beta_{sa}(B) = 1 - \beta_{sa}(B)$. Therefore, at non-zero lags, the magnitude effect of Θ on the coefficients is the same.

Figures 12–15 show the calendar month seasonal factor estimates from the International Airline Passenger data from the filters determined by small, intermediate and large values of Θ always with $\theta = 0.6$. The coefficient values were specified as fixed in X-13ARIMA-SEATS and therefore not estimated. They are not influenced by the data. The seasonal factors they produce reveal the strong influence of Θ. For the factors from $\Theta = 0.0$ and 0.3, there are frequent rapid changes. For the airline series, whose estimates are $\left(\hat{\theta}, \hat{\Theta}\right) = (0.4, 0.6)$, especially the imposed use of $\Theta = 0.0$, results in excessive smoothing of the seasonal adjustment, leading to excessively large revisions (not shown). For $\Theta = 0.9$ the seasonal factors are effectively fixed and not locally adaptive.

[13] Θ has influence on the trend estimate and θ has some on the seasonal estimate, see (6.9)–(6.11) of Hillmer and Tiao (1982). The magnitude of the nonseasonal coefficient θ can be shown to strongly influence trend estimates.

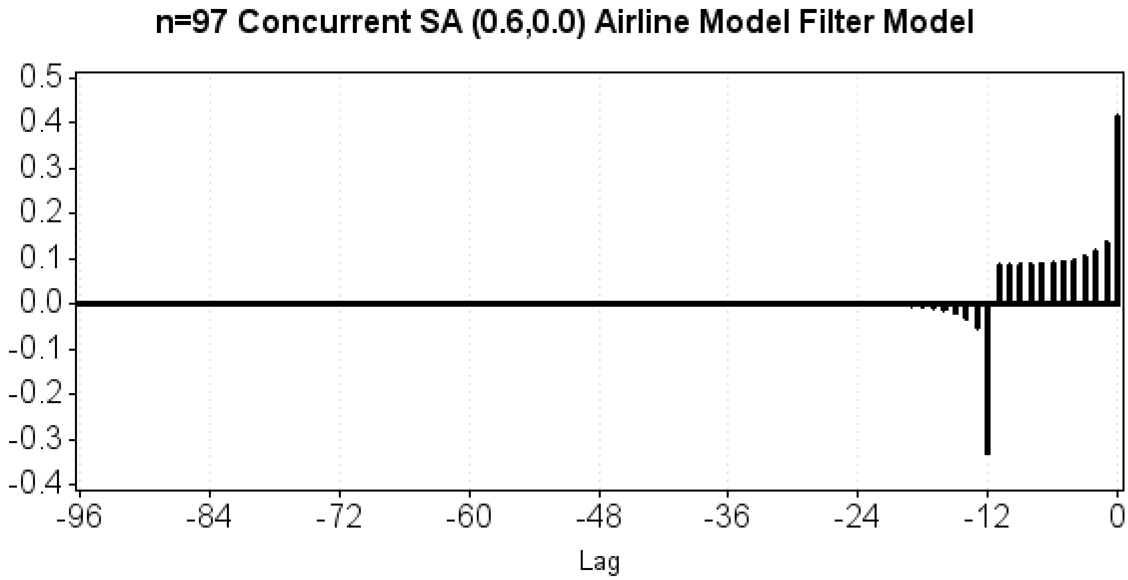

Figure 8: With $\theta = 0.6$ and $\Theta = 0.0$, the coefficients become negligible after one year, with largest-magnitude values at the seasonal lag. Hence large SA revisions are possible when sa_t is re-estimated from an additonal year of data.

12 Autocorrelation and Smoothness Properties of Estimates

12.1 Simplistic Autocorrelation Comparison Criteria for Smoothness and Nonsmoothness

We begin our autocorrelation-based consideration of the smoothing properties of estimates. Some simplistic definitions will support the exposition. When comparing two stationary series X_t and Y_t, we say that X_t is *smooth* if $\rho_1^X > 0$. Further, X_t is *smoother than* Y_t *when differences of scale are accounted for* (a qualification we will omit for brevity after an illustrative example), when the autocorrelations satisfy $\rho_1^X > \rho_1^Y$. The series Y_t is *nonsmooth* if $\rho_1^Y < 0$. A smooth series is therefore smoother than all white noise series and all nonsmooth series. If Y_t is nonsmooth and $\rho_1^Y < \rho_1^X$ holds, then Y_t is *more nonsmooth* than X_t. A nonsmooth series is thus more nonsmooth than all white noise series and all smooth series.

We start with intermediate-time SAR(1) estimates. Similar results for differenced airline model decompositions with component estimates from bi-infinite data are obtained in Section 12.5. Most often, the series considered are calendar month series, so these autocorrelations are seasonal autocorrelations in the time scale of the observations Z_t.

12.2 SAR(1): Autocorrelations of $\hat{N}_t$ and Reduced Smoothness Relative to Z_t

By direct calculation from (33) or from the seasonal MA(1) model (76) derived for intermediate-time $\hat{N}_t$ in Subsection 7.4, the autocorrelations of intermediate-time $\hat{N}_t$ are

$$\rho_j^{\hat{N}} = \begin{cases} -\Phi\left(1+\Phi^2\right)^{-1}, & j = q, \\ 0 & j \neq q. \end{cases} \tag{96}$$

In particular, $\rho_q^{\hat{N}}$ is smaller in magnitude than $\rho_q = \Phi$ and opposite in sign. Thus the calendar month subseries of $\hat{N}_t$, yearly series for which $\rho_q^{\hat{N}}$ is the lag one autocorrelation, are nonsmooth. A calendar month

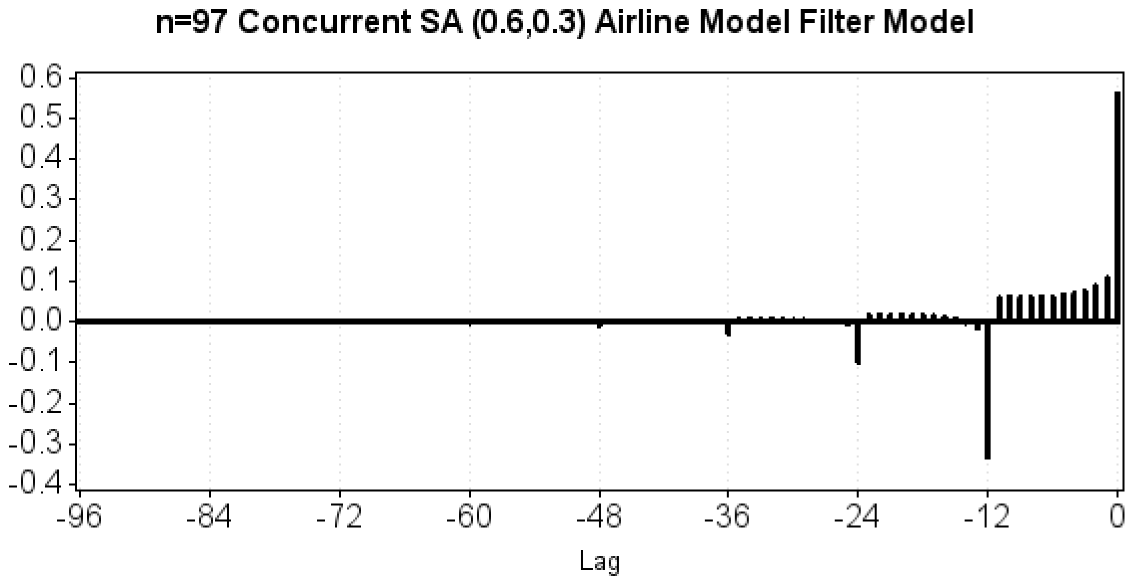

Figure 9: With $\theta = 0.6$ and $\Theta = 0.3$, the effective length of the filter is less than three years. It is quite adaptive but less so than the filter of Figure 8 and therefore subject to smaller revisions.

subseries of Z_t will have lag one autocorrelation $\rho_q = \Phi > 0$ and is therefore smooth. Of course, the calendar month subseries of $\hat{N}_t$ are less variable in the sense that, from (76), $\gamma_0^{\hat{N}} = (1+\Phi)^{-4}\left(1+\Phi^2\right)\sigma_a^2 = (1-\Phi)\left(1+\Phi^2\right)(1+\Phi)^{-3}\gamma_0$ is less than γ_0. Consequently, the scale of the $\hat{N}_t$ is related to that of the Z_t through

$$\sqrt{\gamma_0^{\hat{N}}} = \sqrt{(1-\Phi)\left(1+\Phi^2\right)(1+\Phi)^{-3}}\sqrt{\gamma_0}. \tag{97}$$

The scale reduction factor $\sqrt{(1-\Phi)\left(1+\Phi^2\right)(1+\Phi)^{-3}}$ is approximately 0.113 for $\Phi = 0.95$. These factors quantify the diminished scales of oscillations about the level value 0 seen for the intermediate years in Figure 3. (For the initial and final years, the smaller scale reduction factor $\sqrt{(1-\Phi)(1+\Phi)^{-3}}$ yields 0.082 for $\Phi = 0.95$). Figure 17 below shows calendar month graphs of the nonsmooth component $\hat{N}_t$ and the scale-reduced Z_t, the latter downscaled to have $\hat{N}_t$'s standard error. $\hat{N}_t$ is visibly less smooth, as expected. The next subsection shows that the calendar month series of $\hat{S}_t$ are smooth (and smoother than the corresponding calendar month series of Z_t). Thus the canonical (estimated) signal plus noise decomposition $Z_t = \hat{S}_t + \hat{N}_t$ can be renamed the *canonical smooth plus nonsmooth decomposition*, with $\hat{N}_t$ being the nonsmooth component, etc.

12.3 SAR(1): Autocorrelations of $\hat{S}_t$ and Increased Smoothness Relative to Z_t

By direct calculation of the seasonal lag autocovariances of $Z_{t-q} + 2Z_t + Z_{t+q}$ and (3), we obtain from (38) that the variance and the nonzero autocovariances of intermediate-time $\hat{S}_t$, at lags kq, $k = 1, 2, \ldots$ are

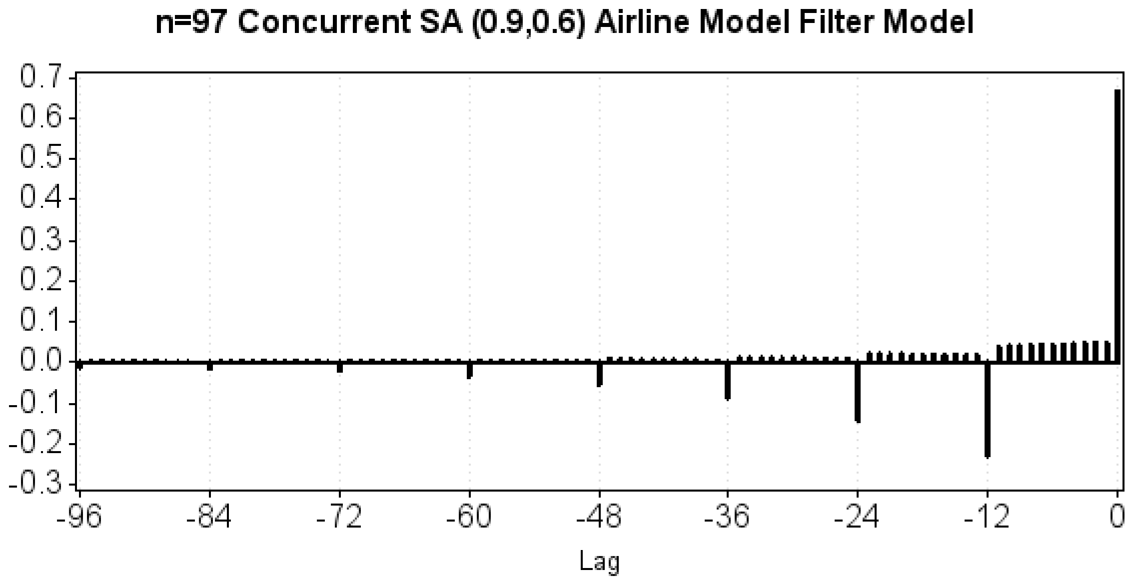

Figure 10: With $\theta = 0.9$ and $\Theta = 0.6$, the effective length of the filter is a year or so longer than in Figure 9 because of the moderately larger-magnitude coefficient values at lags 24 and 36. These could moderately reduce revisions from use of $Z_{n+12k}, k = 1, 2, 3$.

$$\begin{aligned}
\gamma_0^{\hat{S}} &= E\hat{S}_t^2 = \left\{\gamma_0 \frac{\Phi^2}{(1+\Phi)^4}\right\} 2(\Phi+3)(1+\Phi) \\
\gamma_q^{\hat{S}} &= E\hat{S}_t\hat{S}_{t+q} = \left\{\gamma_0 \frac{\Phi^2}{(1+\Phi)^4}\right\} \left(4+3\Phi+\Phi^2\right)(1+\Phi) \\
\gamma_{2q}^{\hat{S}} &= E\hat{S}_t\hat{S}_{t+2q} = \left\{\gamma_0 \frac{\Phi^2}{(1+\Phi)^4}\right\} (1+\Phi)^4 \\
\gamma_{kq}^{\hat{S}} &= \Phi^{k-2}\gamma_{2q}^{\hat{S}}, k \geq 3.
\end{aligned} \tag{98}$$

Division by $\gamma_0^{\hat{S}}$ yields the intermediate-time autocorrelations (99).

$$\rho_j^{\hat{S}} = \begin{cases} \frac{1}{2}\left(4+\Phi(3+\Phi)\right)(\Phi+3)^{-1}, & |j| = q \\ \frac{1}{2}(1+\Phi)^3(\Phi+3)^{-1} & |j| = 2q, \\ \Phi^{k-2}\rho_{2q}^{\hat{S}}, & |j| = kq, \quad k \geq 3, \\ 0, & |j| \neq 0, kq. \end{cases} \tag{99}$$

Using $(3+\Phi)^{-1} > 1/4$ and $\frac{1}{2}(1+\Phi) > \Phi$ for $0 < \Phi < 1$, one readily obtains from (99), (15) and (3) that the nonzero intermediate-time autocorrelations of $\hat{S}_t$ dominate those of S_t, which dominate those of Z_t,

$$\rho_{kq}^{\hat{S}} > \rho_{kq}^{S} > \Phi^k > 0, \; k \geq 1. \tag{100}$$

Same-calendar-month values of Z_t, S_t and $\hat{S}_t$ have positive autocorrelations at all lags, always greatest for

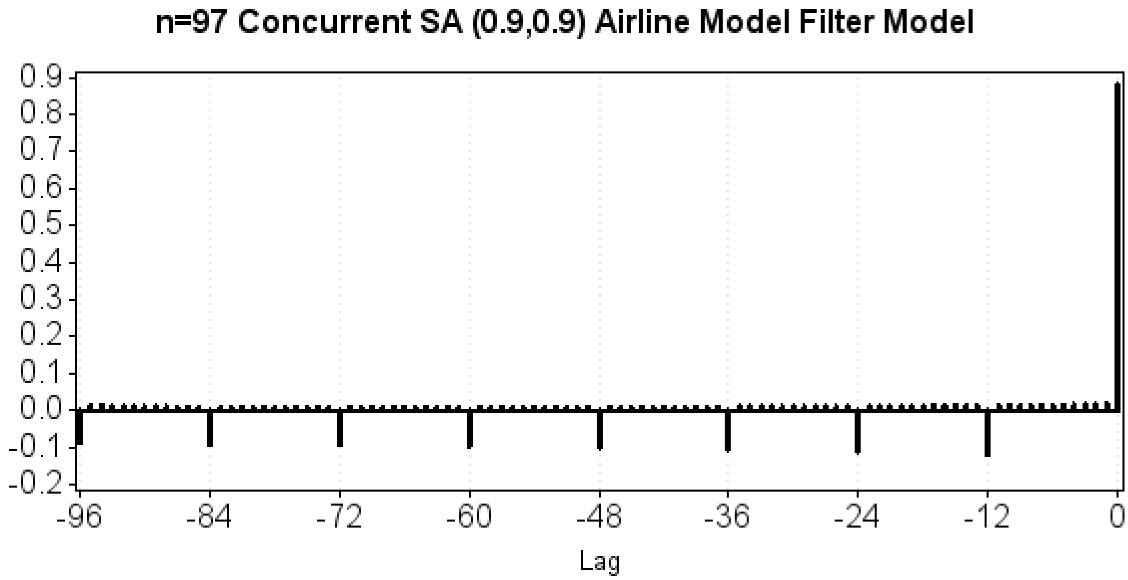

Figure 11: When $\theta = 0.9$ and $\Theta = 0.9$, the effective length of the filter is the length of the data span (also for somewhat larger $n > 97$, it can be shown), with same-calendar-month values persistently most influential. Thus the filter resists domination by rapid changes in a year or two of Z_t values, even close to time $t = n$. On the other hand, revisions from use of $Z_{n+12}, Z_{n=24}, \ldots$ will have little tendency to diminish in size over time and could cumulatively be quite large.

$\hat{S}_t$ and least for Z_t. By calendar month, the $\hat{S}_t$ will evolve more smoothly than Z_t, as Figure 4 illustrates[14]. The largest difference, $\rho_q^{\hat{S}} - \rho_q = \rho_q^{\hat{S}} - \Phi = 2\left(\Phi + 3\right)^{-1} - \frac{1}{2}\Phi$, is approximately 0.031 when $\Phi = 0.95$.

[14] In constrast to the $\hat{N}_t$ smoothness result of Subsection 12.2, we note that the scale of $\hat{S}_t$ is only slightly smaller than the scale of Z_t. From (98), we have $\sqrt{\gamma_0^{\hat{S}}/\gamma_0} \doteq 0.981$ for $\Phi = 0.95$.

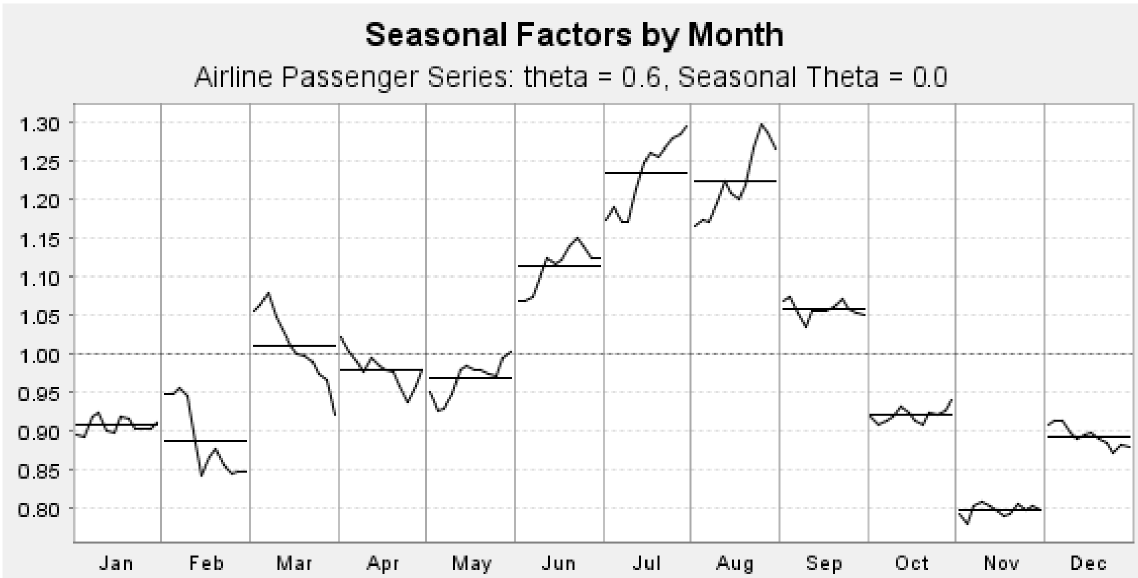

Figure 12: Having $\Theta = 0.0$ results in erratic movements in the seasonal factors estimated from the Airline Passenger series data due to close tracking of detrended series movements, which leads to strong smoothing and a potential for large revisions.

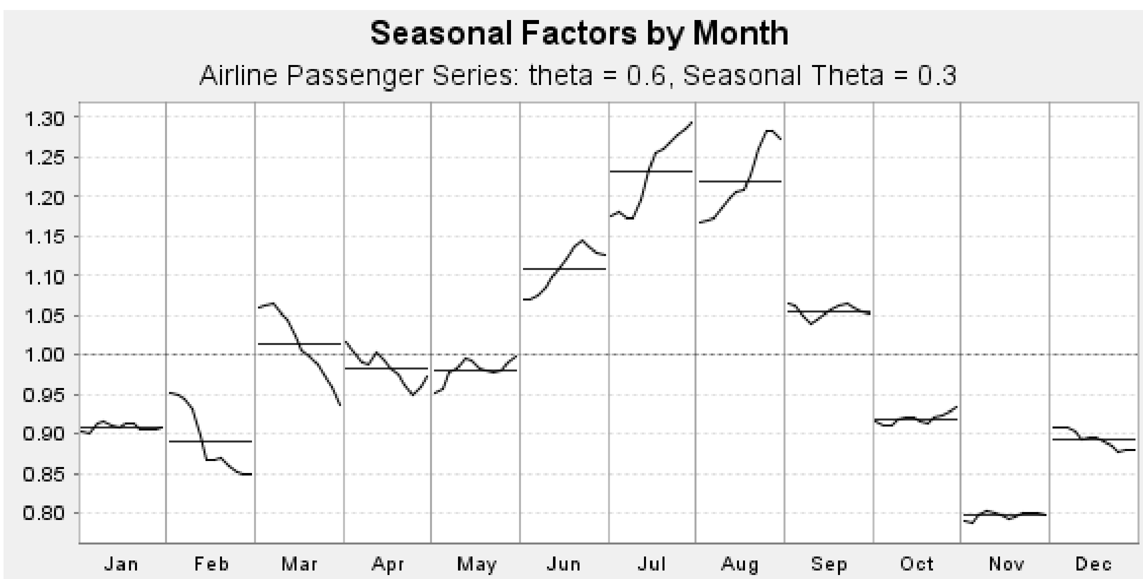

Figure 13: As the discussions of filters and graphs above suggest, when $\Theta = 0.3$, the seasonal factor estimates are very localized and therefore quite variable over the 12 years of the Airline Passenger series data.

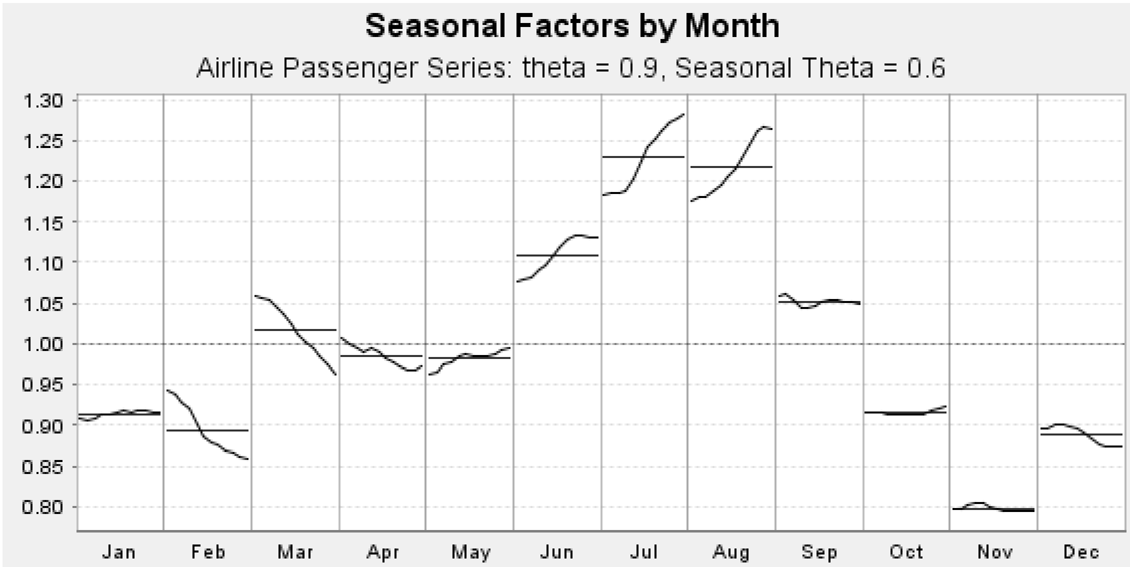

Figure 14: When $\Theta = 0.6$, the seasonal factor estimates from the Airline Passenger series change somewhat less over the 12 years than with $\Theta = 0.3$.

12.4 Stationary Transforms of the Simple SARIMA's Components

As indicated above, with three-component decompositions from an ARIMA Z_t, the differencing operator of Z_t's model is a product, $\delta(B) = \delta_s(B)\,\delta_n(B)$, in which $\delta_s(B)$ is the differencing operator of the seasonal component s_t (from its model or pseudo-s.d.) and $\delta_n(B)$ is the differencing operator of p_t and therefore also of the nonseasonal component $n_t = p_t + u_t$. Rewriting the decomposition (103) as

$$\delta(B)\,Z_t = \delta_n(B)\,\delta_s(B)\,\hat{s}_t + \delta_s(B)\,\delta_n(B)\,\hat{p}_t + \delta(B)\,\hat{u}_t$$

makes clear how each estimate is being overdifferenced: $\delta_s(B)\,\hat{s}_t$, $\delta_n(B)\,\hat{p}_t$, $\delta_n(B)\,\widehat{sa}_t$ and $\hat{u}_t$ are already stationary. In SEATS' output, each of these correctly (minimally) differenced estimates is called the **stationary transformation** of its estimate. The *stationary transform* of each unobserved component s_t, p_t, etc. is defined analogously. SEATS has diagnostics, illustrated in Maravall and Pérez (2012), that test whether or not the theoretical correlations and cross-correlations of these stationary transforms differ in a statistically significant way from simple sample-moment correlation or cross-correlation estimates calculated from the software's numerical decomposition component estimates.

This subsection illustrates calculations of the theoretical autocorrelations and cross-correlations for the stationary transforms of decomposition estimates of the simple ARIMA model (82). For this model, $\delta_s(B) = 1 + B$ and $\delta_n(B) = 1 - B$.

Smoothness results for the stationary transforms are obtained from the theoretical autocorrelation values. For the canonical irregular's intermediate time estimate $\hat{u}_t = \beta_u(B)\,Z_t$ of the model (82) with seasonal period $q = 2$ observations per year, it follows from (89) and (82) that

$$\hat{u}_t = \frac{1}{8}\left(-Z_{t-2} + 2Z_t - Z_{t+2}\right) = \frac{1}{8}\left(-\left(Z_{t+2} - Z_t\right) + \left(Z_t - Z_{t-2}\right)\right) = \frac{1}{8}\left(-a_{t+2} + a_t\right). \tag{101}$$

Thus the only nonzero autocorrelation is $\rho_2^{\hat{u}} = -1/2$.

A conceptually important result is the demonstration that, in the ARIMA case, the analogue of (108) fails for the stationary transforms. That is, $\delta_s(B)\,\hat{s}_t$ and $\delta_n(B)\,\hat{n}_t$ need not be positively correlated: for

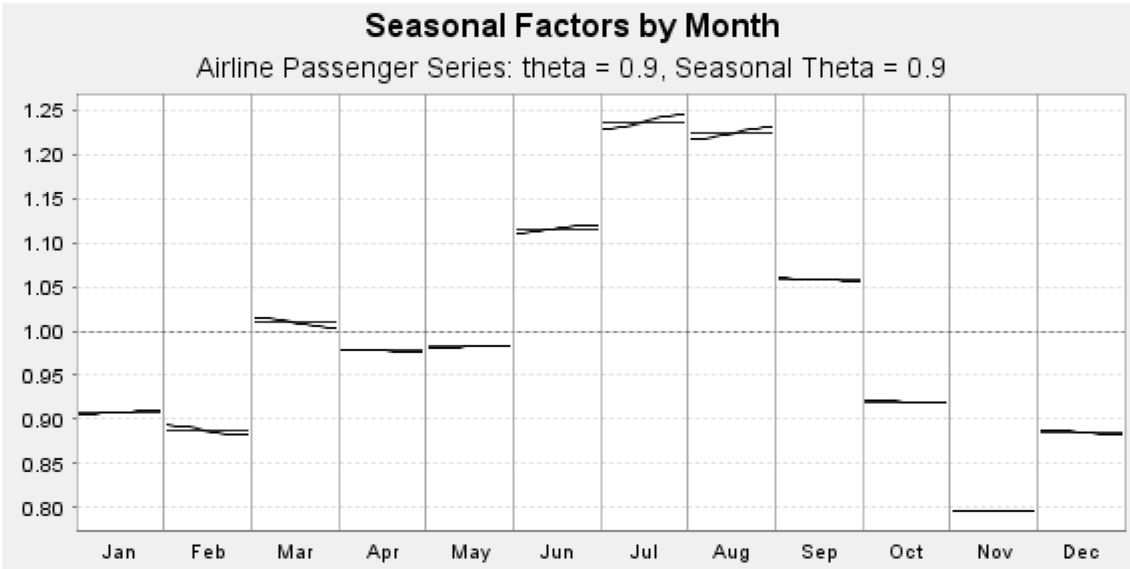

Figure 15: With $\Theta = 0.9$, the calendar month seasonal factor estimates from the Airline Passenger series are almost fixed across the 12 years.

(82), we will obtain (102) for lag zero. We also verify positive cross-correlations for some nonzero lags, and negative cross-correlations for others.

Calculating as in (101), we have

$$\begin{aligned}
(1+B)\,\hat{s}_t &= \frac{1}{16}B^{-2}(1-B)^3 a_t = \frac{1}{16}\left\{a_{t+2} - 3a_{t+1} + 3a_t - a_{t-1}\right\},\\
(1-B)\,\hat{p}_t &= \frac{1}{16}B^{-2}(1+B)^3 a_t = \frac{1}{16}\left\{a_{t+2} + 3a_{t+1} + 3a_t + a_{t-1}\right\},\\
\hat{u}_t &= \frac{1}{8}\left\{-a_{t+2} + a_t\right\},\\
(1-B)\,\hat{u}_t &= \frac{1}{16}\left\{-2a_{t+2} + 2a_{t+1} + 2a_t - 2a_{t-1}\right\},\\
(1-B)\,\widehat{sa}_t &= (1-B)\left(\hat{p}_t + \hat{u}_t\right) = \frac{1}{16}\left\{-a_{t+2} + 5a_{t+1} + 5a_t - a_{t-1}\right\}.
\end{aligned}$$

Following SEATS, we express auto- and cross-covariance results in units of σ_a^2, so $\sigma_a^2 = 1$. It is immediate from the first and last formulas that the lag zero cross-covariance of the stationary transforms of the seasonal and nonseasonal components is zero,

$$E\left(\left\{(1+B)\,\hat{s}_t\right\}\left\{(1-B)\,\widehat{sa}_t\right\}\right) = 0, \tag{102}$$

in contrast to positive value always obtained with stationary Z_t. In addition, the cross-covariances $E\left(\left\{(1+B)\,\hat{s}_t\right\}\left\{(1-B)\,\widehat{sa}_{t-j}\right\}\right), j = \pm 1, \pm 2, \pm 3$ are nonzero. For example, $E(\left\{(1+B)\,\hat{s}_t\right\}\left\{(1-B)\,\widehat{sa}_{t-1}\right\})$ $= 13/16^2$ and $E\left(\left\{(1+B)\,\hat{s}_t\right\}\left\{(1-B)\,\widehat{sa}_{t-2}\right\}\right) = -8/16^2$. Some further lag 0 cross-covariance values are $E\left\{(1+B)\,\hat{s}_t\hat{u}_t\right\} = E\,(1-B)\,\hat{p}_t\hat{u}_t = 4/16^2$.

SEATS does not yet provide the theoretical model-based cross-correlations between the stationary transforms of $\hat{s}_t$ and $\widehat{sa}_t$, so it will not replicate (102). Its method for calculating cross-correlations is shown in the Appendix of Maravall (1994). For components other than sa_t, Table I of Maravall and Pierce (1987) provides lag 1–3 autocorrelation results for the stationary transforms of the components of the simple model

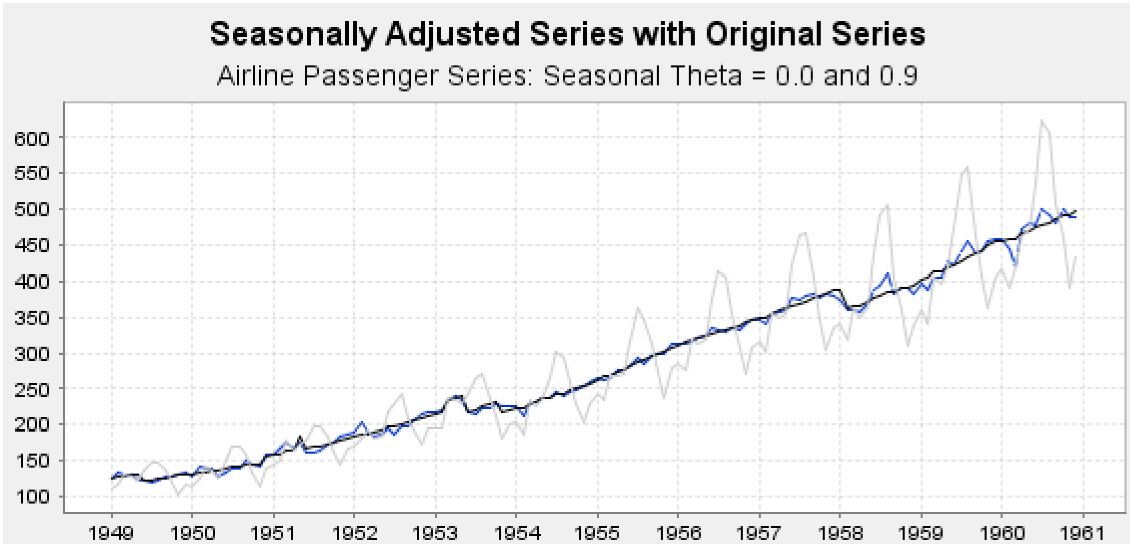

Figure 16: Two extremes. The very smooth, trend-like seasonal adjustment of the Airline Passenger series shown is obtained from division by the volatile $\Theta = 0.0$ seasonal factors of Figure 8. The text of Figure 11 explains why initial revisions from future data are likely to be large. By contrast, the nearly stable $\Theta = 0.90$ same-calendar-month factors of Figure 11 produce a much less smooth adjustment, with residual seasonality visible in the later years.

(82) and their estimates. As typographic errors, the table omits the minus signs of the four autocorrelations of stationary transforms of estimates that require them. Below is a corrected and expanded table, showing the minus signs, corrections of two ρ_1 values, and the autocorrelations of the stationary transforms of sa_t and its estimator $\widehat{sa}_t$. The results for the components were obtained in Subsection 8.1.1. The rest follow from the formulas of the stationary transforms of the estimates of this subsection.

Nonzero Autocorrelations of the Stationary Transforms of the Unobserved and Estimated Decompositions

Transforms	Variance in units of σ_a^2	ρ_1	ρ_2	ρ_3
$(1+B)\,s_t$	$\gamma_0^{(1+B)s} = \frac{1}{8} = 0.125$	-0.50	$-$	$-$
$(1+B)\,\hat{s}_t$	$\gamma_0^{(1+B)\hat{s}} = \frac{20}{16^2} \simeq 0.078$	-0.75	0.30	-0.05
$(1-B)\,p_t$	$\gamma_0^{(1-B)p} = \frac{1}{8} = 0.125$	0.50	$-$	$-$
$(1-B)\,\hat{p}_t$	$\gamma_0^{(1-B)\hat{p}} = \frac{20}{16^2} \simeq 0.078$	0.75	0.30	0.05
$(1-B)\,sa_t$	$\gamma_0^{(1-B)sa} = \frac{3}{8} = 0.375$	$-\frac{1}{6} \simeq -0.167$	$-$	$-$
$(1-B)\,\widehat{sa}_t$	$\gamma_0^{(1-B)\widehat{sa}} = \frac{52}{16^2} \simeq 0.203.$	$\frac{15}{52} \simeq 0.288$	$-\frac{10}{52} \simeq -0.192$	$\frac{1}{52} \simeq 0.019$
u_t	$\gamma_0 = \frac{1}{8} = 0.125$	$-$	$-$	$-$
$\hat{u}_t$	$\gamma_0^{\hat{u}} = \frac{1}{32} \simeq 0.031$	$-$	-0.50	$-$

Applying the smoothness and nonsmoothness criteria of Section 12 to the stationary transforms, this table shows that (i) for $(1+B)\,\hat{s}_t$, the calendar month subseries are smooth, but the series itself is nonsmooth; (ii) for $(1-B)\,\hat{p}_t$, both the series and its calendar month subseries are smooth; (iii) for $(1-B)\,\widehat{sa}_t$, the series is smooth, but its calendar month series are nonsmooth, yet not as nonsmooth as the calendar month subseries of $\hat{u}_t$.

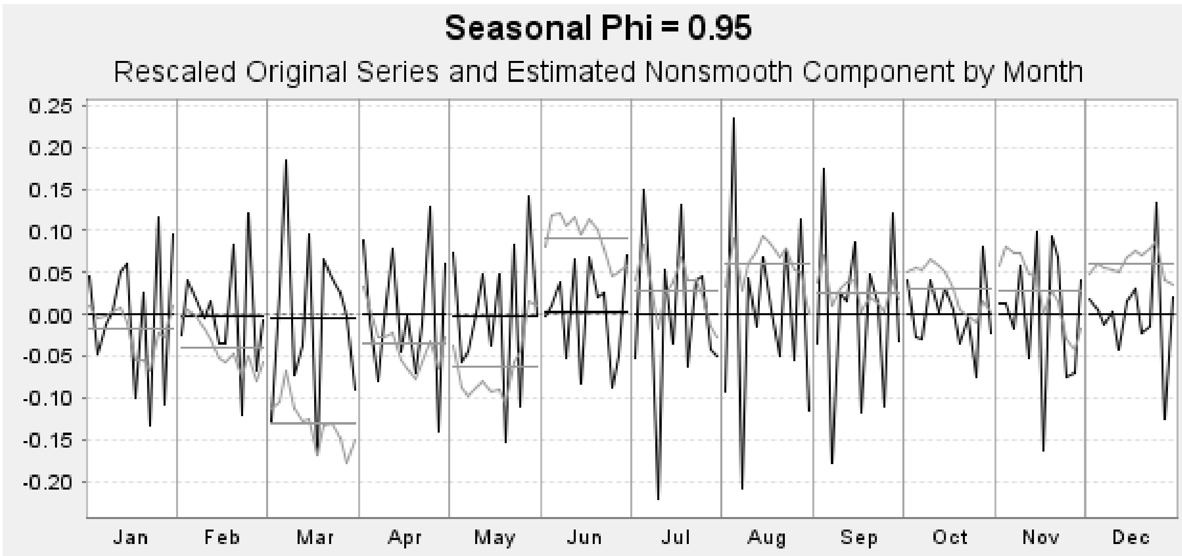

Figure 17: Calendar month plots of the intermediate-time nonsmooth factor estimates $\hat{N}_t$ (darker line) and the rescaled Z_t, downscaled to have the same standard deviation as the $\hat{N}_t$, for $\Phi = 0.95$. The horizontal lines are the calendar month averages of the rescaled Z_t. As the lag 12 autocorrelation analysis suggested, the $\hat{N}_t$ calendar month subseries are visually less smooth than the rescaled Z_t calendar month subseries.

12.5 Airline Model Estimates: Relative Smoothness after Full Differencing

To indicate relative smoothness properties using autocorrelations and variances as above, we must have a stationary decomposition. For monthly series from (93), which has $\delta(B) = (1-B)\left(1-B^{12}\right)$, we examine the fully differenced decomposition

$$\delta(B) Z_t = \delta(B)\hat{s}_t + \delta(B)\hat{p}_t + \delta(B)\hat{u}_t, \tag{103}$$

whose components are the MMSE bi-infinite-data estimates for the components of $\delta(B) Z_t = \delta(B) s_t + \delta(B) p_t + \delta(B) u_t$. We have

$$\rho_{12}^{\delta(B)Z}(\Theta,\theta) = -\Theta\left(1+\Theta^2\right)^{-1}, \quad \rho_{12k}^{\delta(B)Z}(\Theta,\theta) = 0,\ k > 1. \tag{104}$$

The component estimates in (103) are overdifferenced, especially $\hat{u}_t$, which is already stationary. For any choices of $-1 < \theta < 1$ and $0 \leq \Theta < 1$, SEATS outputs the coefficients of the ARIMA or ARMA models of estimators $\hat{s}_t$, $\widehat{sa}_t = Z_t - \hat{s}_t$, $\hat{p}_t$, and $\hat{u}_t$, with innovation variances given in units of σ_a^2. With this information, W-K formulas can be used to obtain models for $\delta(B)\hat{s}_t$, $\delta(B)\hat{p}_t$, and $\delta(B)\hat{u}_t$ From these models, the autocorrelations needed for smoothness analysis like those presented below can be calculated. The simplest model, that of $\delta(B)\hat{u}_t$, is derived in Appendix B as an illustration.

12.5.1 Seasonal Autocorrelations of $\delta(B) Z_t$ and Its Component Estimates for Various θ,Θ

Results are presented in the Tables 2a and beyond for comparison with the autocorrelations of $\delta(B) Z_t = a_t - \theta a_{t-1} - \Theta a_{t-12} + \theta\Theta a_{t-13}$ in Table 1.

Here is a summary of the tabled seasonal lag results. Tables $2a - 2c$ show that, in contrast to $\delta(B) Z_t$, the series $\delta(B)\hat{s}_t$ from the seasonal estimates $\hat{s}_t$ is positively correlated at all seasonal lags considered, 12, 24 and 36, often strongly, indicating that the calendar month subseries of $\delta(B)\hat{s}_t$ will be often be substantially smoother than $\delta(B) Z_t$. Table 3 shows that the opposite is the case for the seasonally adjusted series $\delta(B)\widehat{sa}_t = \delta(B) Z_t - \delta(B)\hat{s}_t$ and for the residual series $\delta(B)\hat{u}_t$. Both have more negative autocorrelations at lag 12 than $\delta(B) Z_t$ and positive autocorrelations at Lag 24 (and negligible autocorrelations at lag 36–not shown). Their calendar month subseries will tend have more changes of direction than $\delta(B) Z_t$.

Table 1. Lag 12 Autocorrelations of $\delta(B) Z_t = (1-\theta B)\left(1-\Theta B^{12}\right) a_t$

$\Theta \backslash \theta$	-0.3	0.0	0.3	0.6	0.9
0.0	0	0	0	0	0
0.3	-0.275	-0.275	-0.275	-0.275	-0.275
0.6	-0.442	-0.442	-0.442	-0.442	-0.442
0.9	-0.497	-0.497	-0.497	-0.497	-0.497

From (104), $\rho_{12}^{\delta(B)Z} < 0$ for $\Theta > 0$, so $\delta(B) Z_t$ will have nonsmooth (calendar month) subseries.

Table 2a. Lag 12 Autocorrelations of $\delta(B)\hat{s}$

$\Theta \backslash \theta$	-0.3	0.0	0.3	0.6	0.9
0.0	0.347	0.467	0.589	0.622	0.222
0.3	0.568	0.644	0.714	0.731	0.481
0.6	0.763	0.803	0.836	0.844	0.715
0.9	0.943	0.952	0.959	0.960	0.931

$\rho_{12}^{\delta(B)\hat{s}} > \left|\rho_{12}^{\delta(B)Z}\right|$ always.

$\delta(B)\hat{s}_t$ has substantially smoother subseries than $\delta(B) Z_t$.

Table 2b. Lag 24 Autocorrelations of $\delta(B)\hat{s}_t$

$\Theta \backslash \theta$	-0.3	0.0	0.3	0.6	0.9
0.0	0.035	0.072	0.121	0.131	0.013
0.3	0.197	0.244	0.294	0.305	0.154
0.6	0.474	0.510	0.545	0.552	0.435
0.9	0.852	0.864	0.873	0.875	0.840

$\rho_{24}^{\delta(B)\hat{s}} > 0 = \rho_{24}^{\delta(B)Z}$. The calendar-month smoothing indicated in Table 2a is reinforced, moderately to strongly at two year's remove.

Table 2c. Lag 36 Autocorrelations of $\delta(B)\hat{s}_t$

$\Theta \backslash \theta$	-0.3	0.0	0.3	0.6	0.9
0.0	$\simeq 0$	$\simeq 0$	$\simeq 0$	$\simeq 0$	$\simeq 0$
0.3	0.059	0.073	0.088	0.092	0.046
0.6	0.284	0.306	0.327	0.331	0.261
0.9	0.767	0.777	0.786	0.788	0.756

$\rho_{36}^{\delta(B)\hat{s}} > 0 = \rho_{36}^{\delta(B)Z}$ for $\Theta \geq 0.3$.

Calendar month smoothing is further reinforced at three years remove, but not as strongly at as two.

Table 3. Lag 12 autocorrelations of $\delta(B)\widehat{sa}_t$

$\Theta\backslash\theta$	-0.3	0.0	0.3	0.6	0.9
0.0	-0.297	-0.465	-0.590	-0.646	-0.659
0.3	-0.520	-0.548	-0.573	-0.586	-0.590
0.6	-0.520	-0.525	-0.529	-0.532	-0.533
0.9	-0.502	-0.502	-0.502	-0.502	-0.502

All $\rho_{12}^{\delta(B)\widehat{sa}} < \rho_{12}^{\delta(B)Z} < 0$.

$\delta(B)\widehat{sa}_t$ is more nonsmooth than $\delta(B)Z_t$.

Table 4. Lag 12 autocorrelations of $\delta(B)\hat{u}$

$\Theta\backslash\theta$	-0.3	0.0	0.3	0.6	0.9
0.0	-0.667	-0.667	-0.667	-0.667	-0.667
0.3	-0.591	-0.591	-0.591	-0.591	-0.591
0.6	-0.533	-0.533	-0.533	-0.533	-0.533
0.9	-0.502	-0.502	-0.502	-0.502	-0.502

$\rho_{12}^{\delta(B)\hat{u}}$ is more negative than $\rho_{12}^{\delta(B)Z}$.

The subseries of $\delta(B)\hat{u}_t$ have a greater tendency than $\delta(B)Z_t$ to change direction. They are more nonsmooth.

12.5.2 Monthly Smoothness Results

One expects the estimated trend to be smooth on the monthly time scale and the estimated irregulars to be nonsmooth on this scale, due to trend removal. We examined the lag 1-12 autocorrelations of the differenced trend estimates, $\delta(B)\hat{p}_t$ for the Θ, θ under consideration. At lag 12, all are negative: at yearly intervals, the differenced trend is nonsmooth. At lags 1–6 all are positive. At the remaining lags 7-11, some or all can be positive and some or all can be negative, depending on (Θ, θ). In summary, $\delta(B)\hat{p}_t$ will have a tendency to move in the same direction for six months, sometimes more, i.e.. a tendency for half-year smoothness. This is in strong contrast to $\delta(B)Z_t$, which, among lags 1–6, has a non-zero autocorrelation only at lag one, $\rho_1^{\delta(B)Z}(\Theta,\theta) = -\theta\left(1+\theta^2\right)^{-1}$. This is negative, indicating nonsmoothness (except when $\theta > 0$). The tabled results below for $\hat{u}_t$ and $\delta(B)\hat{u}_t$ indicate nonsmoothness, with $\delta(B)\hat{u}_t$ being more nonsmooth than $\delta(B)Z_t$.

Table 5. Lag 1 Autocorrelations of $\hat{u}_t$

$\Theta\backslash\theta$	-0.3	0.0	0.3	0.6	0.9
0.0	-0.650	-0.500	-0.350	-0.200	-0.034
0.3	-0.650	-0.500	-0.350	-0.200	-0.040
0.6	-0.650	-0.500	-0.350	-0.200	-0.042
0.9	-0.650	-0.500	-0.350	-0.200	-0.048

$\rho_1^{\delta(B)\hat{u}} < 0$ always: Monthly $\hat{u}_t$ are nonsmooth.

Table 6. Lag 1 Autocorrelations $\rho_1^{\delta(B)Z}(\Theta,\theta) = -\theta\left(1+\theta^2\right)^{-1}$ of $\delta(B)Z_t$

$\Theta\backslash\theta$	-0.3	0.0	0.3	0.6	0.9
0.0	0.275	0.0	-0.275	-0.441	-0.497
0.3	0.275	0.0	-0.275	-0.441	-0.497
0.6	0.275	0.0	-0.275	-0.441	-0.497
0.9	0.275	0.0	-0.275	-0.441	-0.497

Monthly $\delta(B)Z_t$ are nonsmooth for $\theta > 0$, smooth for $\theta < 0$.

Table 7. Lag 1 Autocorrelations of $\delta(B)\hat{u}_t$

$\Theta\backslash\theta$	-0.3	0.0	0.3	0.6	0.9
0.0	-0.756	-2/3	-0.591	-0.533	-0.502
0.3	-0.756	-2/3	-0.591	-0.533	-0.502
0.6	-0.756	-2/3	-0.591	-0.533	-0.502
0.9	-0.756	-2/3	-0.591	-0.533	-0.502

$\rho_1^{\delta(B)\hat{u}} < \min\left(0, \rho_1^{\delta(B)Z}\right)$. $\delta(B)\hat{u}_t$ is always more nonsmooth than $\delta(B)Z$.

Also $\rho_2^{\delta(B)\hat{u}} > 0$ in all cases. (Not tabled.)

13 Concluding Remarks

The simple seasonal models focussed on have provided very informative and tractable formulas for two- and three-component decompositions of seasonal time series. The estimates' auto- and cross-correlation formulas have led to new insights and results. For example, the common finding of negative sample autocorrelations, often at the first lag and almost always at the first seasonal lag of the estimated irregular component $\hat{u}$ (or differenced $\hat{u}$) can now be both anticipated and understood from the smoothing results, in combination with the knowledge that $\hat{u}$ can be regarded both as the detrended version of the seasonally adjusted series $Z - \hat{s}$ and also as the deseasonalized version of the detrended series $Z - \hat{p}$. Thus one expects $\hat{u}$ to be nonsmooth at both the monthly scale where (differenced) $\hat{p}$ is smooth and also at the yearly scale of calendar month series where (differenced) $\hat{s}$ is smooth. Similarly for the finding of negative sample autocorrelations at the seasonal lag of the differenced seasonally adjusted series. This result now appears as an inevitable result of removing a seasonal component whose calendar month subseries are smooth, and not as a defect of the seasonal adjustment procedure, contrary to a view expressed in some of the literature motivating McElroy (2012). Here it should be noted that because seasonal adjustment nonsmoothness occurs on the yearly time scale of calendar month series, not on the monthly time scale of real-time economic analysis, it may have no practical importance.

The capacity to provide illuminating precise answers to many questions is a valuable feature of ARIMA-model-based seasonal adjustment. So too are the uncertainty measures (not accounting for sampling error) that AMBSA can provide for additive direct seasonal adjustments and their period-to-period changes and for growth rates from log-additive adjustments, information not available from current X-11 implementations. But AMBSA results depend completely upon the determination of an adequately fitting, admissible regARIMA model, either by the software's automatic modeling procedure or by the software user. This can be a major challenge especially with a long data span, because of the time-varying and sampling-based nature of economic indicators.

This situation, and the quite variable time series backgrounds of people tasked with seasonal adjustment, underscore the value of software like X-13ARIMA-SEATS and JDemetra+ which offer both ARIMA model-based and X-11-ARIMA method seasonal adjustments. The latter depend on the model only for a year or so of forecasts and thus are less dependent on the model's quality or on the user's resourcefulness when faced with poor diagnostics for an important series. Both methods produce similar seasonal adjustments quite often. This is not so surprising because their filters can be quite close, see Depoutot and Planas (1998) and Chu, Tiao and Bell (2012). Analysis of substantial differences, when they occur, can suggest that one is to be preferred, or can suggest changes in software options that lead to a third more satisfactory adjustment. There is no conceptual problem with adjusting some series in a given category with one of these well-established methods and adjusting the category's other series with the other method. This is already a practice in some statistical agencies. The goal is to have adjustments (from plausible adjustment factors) with no detectable residual seasonality and without an excessive number of large revisions when an additional year or so of future data is added to the time series.

Disclaimer. Results of ongoing research are provided to inform interested parties and stimulate discussion. Any opinions expressed are those of the authors and not necessarily those of the U.S. Census Bureau or the Bank of Spain.

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14 Appendix A: Derivations of Wiener-Kolmogorov Formulas and Cross-Covariances

An MMSE linear estimate is characterized by the property that its error at any time t is uncorrelated with all of the estimation data Z_τ, see Wikipedia (2013). For the regression formula (21), observe that $E\left(S - \Sigma_{SS}\Sigma_{ZZ}^{-1}Z\right)Z' = ESZ' - \Sigma_{SS} = \Sigma_{SS} - \Sigma_{SS} = 0$, the $n \times n$ zero matrix. Thus, with bi-infinite data Z_τ, $\tau = 0, \pm 1, \pm 2, \ldots$, the filter $\beta_S(B) = \sum_{j=-\infty}^{\infty} \beta_j^S B^j$ with MMSE $\hat{S}_t = \beta_S(B) Z_t = \sum_{j=-\infty}^{\infty} \beta_j^S Z_{t-j}$ is characterized by

$$
\begin{aligned}
0 &= E\left(S_t - \hat{S}_t\right) Z_{t-k} = ES_t\left(S_{t-k} + N_{t-k}\right) - \sum_{j=-\infty}^{\infty} \beta_j^S EZ_{t-j}Z_{t-k} &(105)\\
&= \gamma_k^S - \sum_{j=-\infty}^{\infty} \beta_j^S \gamma_{k-j}^Z, k = 0, \pm 1, \pm 2, \ldots \quad . &(106)
\end{aligned}
$$

Multiplying (106) by $e^{i2\pi k\lambda} = e^{i2\pi j\lambda} e^{i2\pi(k-j)\lambda}$ and summing over $-\infty < k < \infty$, one obtains from (105)–(106), after interchanging the order of summation, that

$$
\begin{aligned}
g_S(\lambda) &= \sum_{k=-\infty}^{\infty} e^{i2\pi k\lambda} \gamma_k^S = \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} \beta_j^S e^{i2\pi k\lambda} \gamma_{k-j}^Z \\
&= \sum_{j=-\infty}^{\infty} \beta_j^S e^{i2\pi j\lambda} \sum_{k=-\infty}^{\infty} \gamma_{k-j}^Z e^{i2\pi(k-j)\lambda} = \beta_S\left(e^{i2\pi\lambda}\right) g(\lambda). &(107)
\end{aligned}
$$

The W-K formula for $\hat{S}_t$ in (66) follows via division by $g(\lambda)$. (The sum over k in (107) has the same value $g(\lambda)$ for every j, so set $j = 0$.) The formula for $\beta_N(B)$ follows from $\beta_N(B) = 1 - \beta_S(B)$.

This derivation is a simplification of the one given in Chapter 5 of Whittle (1963) of the more general formulas for the case in which the signal and noise series, S_t and N_t, are stationarily cross-correlated. For this case, McElroy and Maravall (2014) provides the ARIMA generalizations.

14.1 Cross-Covariance Properties of Stationary-Case Estimates

Among other results, we now verify (77) and show that stationary-case estimates are positively correlated at lag zero,

$$E\hat{S}_t\hat{N}_t > 0. \tag{108}$$

To obtain (77) and a useful alternative interpretation of $g_\epsilon(\lambda)$, we note from (105) that $\hat{S}_t$ and $\epsilon_t = S_t - \hat{S}_t$ on right hand side of $S_t = \hat{S}_t + (S_t - \hat{S}_t) = \hat{S}_t + \epsilon_t$ are uncorrelated. From this, an autocovariance decomposition and therefore also the spectral density decomposition of S_t follows, $g_S(\lambda) = g_{\hat{S}}(\lambda) + g_\epsilon(\lambda)$. Thus, from (74),

$$g_\epsilon(\lambda) = g_S(\lambda) - \frac{g_S(\lambda)^2}{g(\lambda)} = g_S(\lambda)\left\{\frac{g(\lambda) - g_S(\lambda)}{g(\lambda)}\right\} = \frac{g_S(\lambda)\, g_N(\lambda)}{g(\lambda)},$$

which is (77). Similarly, for any j, $N_{t-j} = \hat{N}_{t-j} - \epsilon_{t-j}$, with which we can now verify the equality of the cross-covariances $\gamma_j^{\hat{S}\hat{N}} = E\hat{S}_t\hat{N}_{t-j}$ and $\gamma_j^\epsilon = E\epsilon_t\epsilon_{t-j}$,

$$\gamma_j^{\hat{S}\hat{N}} = \gamma_j^\epsilon, j = 0, \pm 1, \ldots \qquad . \tag{109}$$

This follows from $0 = ES_tN_{t-j} = E\hat{S}_t\hat{N}_{t-j} - E\epsilon_t\epsilon_{t-j} - E\hat{S}_t\epsilon_{t-j} + E\hat{N}_{t-j}\epsilon_t$, whose last two terms are zero by (105). Finally, because $g_\epsilon(\lambda)$ is positive except at the finitely many λ where $g_S(\lambda)$ or $g_N(\lambda)$ is zero, its integral, which is equal to γ_0^ϵ, is positive. From this (108) follows. Further, (109) shows that the cross-spectral density $g_{\hat{S}\hat{N}}(\lambda) = \sum_{j=-\infty}^{\infty} \gamma_j^{\hat{S}\hat{N}} e^{i2\pi j\lambda}$ coincides with $g_\epsilon(\lambda)$. This shows that there are bi-infinite estimation analogues of all of the matrix formulas (28)-(29).

15 Appendix B: W-K Derivation of the SARMA Model of $\delta(B)\,\hat{u}_t$ for the Airline Model

The W-K estimate $\hat{u}_t$ of the canonical airline model decomposition's irregular component u_t has the pseudo-s.d.

$$g_{\hat{u}}(\lambda) = \frac{g_u^2(\lambda)}{g(\lambda)} = \frac{\sigma_u^4}{\sigma_a^2}\frac{\left|\delta\left(e^{i2\pi\lambda}\right)\right|^2}{\left|\vartheta\left(e^{i2\pi\lambda}\right)\right|^2},$$

with $g_u(\lambda) = \sigma_u^2$. Thus, from (93), $\hat{u}_t$ has the stationary noninvertible seasonal ARMA(1,1)(1,1)$_{12}$ model

$$(1 - \theta B)\left(1 - \Theta B^{12}\right)\hat{u}_t = (1 - B)\left(1 - B^{12}\right)c_t,$$

with white noise c_t having variance σ_u^4/σ_a^2. Similarly, from (60), $\hat{J}_t = \delta(B)\,\hat{u}_t$, the fully differenced $\hat{u}_t$ has s.d.

$$g_{\hat{J}}(\lambda) = \left|\delta\left(e^{i2\pi\lambda}\right)\right|^2 g_{\hat{u}}(\lambda) = \frac{\sigma_u^4}{\sigma_a^2}\frac{\left|\delta\left(e^{i2\pi\lambda}\right)\right|^4}{\left|\vartheta\left(e^{i2\pi\lambda}\right)\right|^2}.$$

So its model is the noninvertible seasonal ARMA(1,2)(1,2)$_{12}$ model
$(1 - \theta B)(1 - \Theta B^q)\,\hat{J}_t = (1 - B)^2\left(1 - B^{12}\right)^2 c_t$. Multiplied out, this model is

$$\left(1 - \theta B - \Theta B^{12} + \theta\Theta B^{13}\right)\hat{J}_t = \left(1 - 2B + B^2 - 2B^{12} + 4B^{13} - 2B^{14} + B^{24} - 2B^{25} + B^{26}\right)c_t. \tag{110}$$

Expanded model formulas like this are what the algorithms for calculating autocovariances referenced in Subsection 12.5.1 require and what SEATS outputs.

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