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A Simple Joint Confidence Region for A Ranking of K Populations: Application to American Community Survey's Travel Time to Work Data

Martin Klein
Tommy Wright
Jerzy Wieczorek [†]

[†] Carnegie Mellon University, Pittsburgh, PA

Center for Statistical Research and Methodology
Research and Methodology Directorate
U.S. Census Bureau
Washington, D.C. 20233

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A Simple Joint Confidence Region for A Ranking of K Populations: Application to American Community Survey's Travel Time to Work Data

Martin Klein, Tommy Wright, and Jerzy Wieczorek

National statistical agencies lack statistical methodology to express uncertainty in their released (explicit or implicit) estimated rankings, and we present a simple and novel measure for them to consider using theory, real data, and a visualization. Ranking a collection of populations from smallest to largest is typically based on the ranking of the observed estimates of parameters of the populations which make use of sample survey data. For example, the U.S. Census Bureau produced an "explicit" ranking of the states based on observed sample estimates during 2011 of estimated mean travel time to work for each state. Lack of statistical theory prevents a direct expression of uncertainty for the estimated ranking. We construct a joint confidence region for the true unknown ranking and present a visualization of the region that makes it easy to see the estimated ranking and its associated uncertainty. The observed estimated ranking is one of many likely rankings revealed by the joint confidence region visualization.

KEY WORDS: Bonferroni Inequality; Joint confidence region for ranking; Official statistics.

1. INTRODUCTION

Rankings (explicit or implicit) of $K \geq 2$ populations or governmental units based on sample survey data are usually released without direct statistical statements of uncertainty on estimated rankings. Our main objective is to provide simple and easy to use statistical methodology for expressing uncertainty in released rankings based on data from sample surveys by statistical agencies. A visualization facilitates communication with wide audiences.

Martin Klein is Principal Researcher, Center for Statistical Research and Methodology (CSRM), U. S. Census Bureau, Washington, D.C. 20233 and adjunct faculty at UMBC (E-mail: martin.klein@census.gov). Tommy Wright is Chief, CSRM, U. S. Census Bureau, Washington, D.C., 20233 and adjunct faculty at Georgetown University (E-mail: tommy.wright@census.gov). Jerzy Wieczorek (formerly with the U. S. Census Bureau) is a graduate student in statistics at Carnegie Mellon University, Pittsburgh, PA (E-mail: jerzy@cmu.edu). The views expressed are those of the authors and not necessarily those of the U. S. Census Bureau.

Formally, assume K populations with associated independent continuous random variables $Y_1, ..., Y_K$ and respective cumulative distribution functions $F_1(y), ..., F_K(y)$. Let θ_k be a real-valued characteristic (parameter) related to $F_k(y)$, for $k = 1, ..., K$. While the values of $\theta_1, ..., \theta_K$ are unknown, it is desired to rank the K populations from smallest to largest based on these unknown values, i.e., based on

$$\theta_{(1)} < \theta_{(2)} < \cdots < \theta_{(k)} < \cdots < \theta_{(K)}. \tag{1}$$

If $Y_{k1}, ..., Y_{kn_k}$ is a probability sample of size n_k from the k^{th} population where the statistic $\hat{\theta}_k = \hat{\theta}_k(Y_{k1}, ..., Y_{kn_k})$ is an estimator of θ_k for $k = 1, ..., K$, we rank the K populations based on the observed ranking of the values, $\hat{\theta}_1, ..., \hat{\theta}_K$, i.e.,

$$\hat{\theta}_{(1)} < \hat{\theta}_{(2)} < \cdots < \hat{\theta}_{(k)} < \cdots < \hat{\theta}_{(K)}. \tag{2}$$

For example, data from the U. S. Census Bureau's American Community Survey (ACS) produced an *explicit ranking* of the $K = 51$ states (including Washington, D.C.) based on observed sample estimates during 2011 of θ_k the mean travel time to work (in minutes) for workers 16 years and over who did not work at home (henceforth "mean travel time to work") for state k, where $k = 1, ..., 51$. Given estimates in a table without an explicit ranking, users tend to compare states looking for smallest or largest estimates and for relative standings among the states. We refer to such tables as motivating *"implicit" rankings*.

Because rankings based on the observed values of $\hat{\theta}_1, ..., \hat{\theta}_K$ can vary due to sampling variability, *widely understood* and *robust* (valid and applicable in many situations) statements of uncertainty should accompany each released ranking.

In this paper, a collection of joint confidence intervals for $\theta_1, ..., \theta_K$ form the basis for the measure presented. Knowledge of the specific complex sampling design and estimation methodology for each population is not required. In Section 2, we present a simple mathematical result. Section 3 uses this mathematical result to provide general theory for constructing a joint confidence region for the overall ranking. Examples using the ACS's travel time to work data are given in Section 4. A simulation study is given in Section 5, and Section 6 gives concluding remarks.

Overview of the American Community Survey. Conducted by the U. S. Census Bureau, the ACS's sampling design is basically a national stratified random sample with sampling and estimation following a finite population design-based framework. Data are collected throughout the year. (See https://www.census.gov/programs-surveys/acs/technical-documentation/code-lists.html.) The ACS provides data every year - giving communities current information needed to plan investments and services. The sample survey generates data that help determine how hundreds of billions of dollars in federal and state funds are distributed each year. Currently, over 3,500,000 housing unit addresses are contacted each year by Internet, mail, telephone, or face-to-face to provide data for statistical estimates at various geographic levels - large and small. In addition to travel time to work, the ACS questionnaire asks about: age, sex, race, family and relationships, income and benefits, health insurance, education, veteran status, disabilities, where you work and how you get there, and where you live and how much you pay for some essentials.

Among "ranking tables" based on many topics using data collected by the American Community Survey for 2011 is Table 1. From Table 1, the 51 states (including Washington, D.C.) are ranked from largest to smallest by estimated mean travel time to work (See https://factfinder.census.gov/bkmk/table/1.0/en/ACS/11 1 YR/R0801.US01PRF.) From the Statistical Significance column, we see the results of 50 separate tests of significance ($\alpha = 0.1$) for Alabama as the selected state with each of the other states. Alabama is not statistically significantly different from Tennessee, Michigan, Nevada, Mississippi, South Carolina, and Rhode Island.

In Table 1, the margin of error gives uncertainty in the estimate for each state separately; and the tests of significance compare one state's estimate with those of each of the other states. However, a direct assessment of the uncertainty in the estimated ranking would involve all of the states simultaneously and their relative standing to each other. To say that Alabama's estimated rank is 25, one includes data from all states. We seek to provide an uncertainty measure directly focused on the overall estimated ranking.

Table 1: Mean Travel Time To Work Of Workers 16 Years And Over Who Did Not Work At Home (Minutes)

Rank	Geographical Area	Statistical Significance?	Estimated Mean	Margin of Error
	United States		25.5	+/-0.1
1	Maryland		32.2	+/-0.2
2	New York		31.5	+/-0.2
3	New Jersey		30.5	+/-0.2
4	District of Columbia		30.1	+/-0.5
5	Illinois		28.2	+/-0.2
6	Massachusetts		28.0	+/-0.2
7	Virginia		27.7	+/-0.2
8	California		27.1	+/-0.1
8	Georgia		27.1	+/-0.3
10	New Hampshire		26.9	+/-0.5
11	Pennsylvania		25.9	+/-0.1
12	Florida		25.8	+/-0.2
13	Hawaii		25.7	+/-0.4
14	West Virginia		25.6	+/-0.5
15	Washington		25.5	+/-0.2
16	Delaware		25.3	+/-0.6
17	Connecticut		25.0	+/-0.3
18	Arizona		24.8	+/-0.2
18	Texas		24.8	+/-0.1
20	Colorado		24.5	+/-0.3
20	Louisiana		24.5	+/-0.2
22	Tennessee	#	24.2	+/-0.2
23	Michigan	#	24.1	+/-0.2
23	Nevada	#	24.1	+/-0.4
25	Alabama	##	23.9	+/-0.2
25	Mississippi	#	23.9	+/-0.4
27	South Carolina	#	23.6	+/-0.3
28	Indiana		23.5	+/-0.2
29	Maine		23.4	+/-0.4
29	North Carolina		23.4	+/-0.2
29	Rhode Island	#	23.4	+/-0.5
32	Missouri		23.1	+/-0.2
32	Ohio		23.1	+/-0.1
34	Minnesota		23.0	+/-0.2
35	Kentucky		22.9	+/-0.2
36	Oregon		22.5	+/-0.3
37	Vermont		21.9	+/-0.5
37	Wisconsin		21.9	+/-0.2
39	Utah		21.6	+/-0.3
40	New Mexico		21.4	+/-0.4
41	Arkansas		21.3	+/-0.4
42	Oklahoma		21.1	+/-0.2
43	Idaho		19.7	+/-0.4
44	Kansas		18.9	+/-0.3
45	Iowa		18.8	+/-0.2
46	Alaska		18.4	+/-0.5
47	Montana		18.2	+/-0.5
48	Nebraska		18.1	+/-0.3
48	Wyoming		18.1	+/-0.8
50	North Dakota		16.9	+/-0.6
50	South Dakota		16.9	+/-0.5

The ## indicates the selected state is being compared with each of the other 50 states.
An # next to a state indicates when an estimate is not statistically significant from the estimate for the selected state (##).
Source: 2011 1-Year American Community Survey, Ranking Table R0801, U. S. Census Bureau. For more information on the ACS, see https://www.census.gov/programs-surveys/acs/.

We highlight a few related papers summarized in Frey (2008). In a seminal paper from the ranking and selection literature, Bechhofer (1954) presents a procedure for computing sample sizes n_k for ranking K populations where the ranking is based on the observed sample means. Assuming the usual Bayesian setup of priors on the parameters θ_k, the focus is on how to go from posteriors on the parameters θ_k to a ranking of the parameters. The literature suggests that ranking on posterior means can lead to "very poor results" (Frey, 2008). Govindarajulu and Harvey (1974) "...point out that simply choosing the ranking with the highest posterior probability may not be an ideal approach, even if it were possible" (Frey, 2008). Louis (1984) argues that any ranking of populations based on θ_k should consider the collection or ensemble $\{\theta_1, \theta_2, ..., \theta_K\}$ and not the θ_k individually. Also see Klein and Wright (2011). Goldstein and Spiegelhalter (1996) suggest the bootstrap as a means of obtaining interval estimates for ranks, as do Hall and Miller (2009) and Wright, Klein, and Wieczorek (2013, 2014, In Press).

The primary objective in this paper is to present a frequentist joint confidence region for the overall ranking whose coverage probability has a guaranteed lower bound. The proposed approach does not require intensive computations.

2. MAIN RESULT

One could imply uncertainty in an estimated ranking (2) through confidence intervals and hypothesis tests for individual parameters θ_k's, and for the pairwise differences $\theta_k - \theta_{k'}$ (e.g., Wright, Klein, and Wieczorek, In Press). This is the approach currently taken by the Census Bureau's American Community Survey and illustrated earlier with Table 1. However, these approaches do not provide a direct measure of uncertainty for the overall estimated ranking. Alternatively, one may consider the individual ranks as the parameters of interest, and inferences can be drawn on them directly. The unknown true ranks are denoted by $r_1, ..., r_K$, and they are defined such that the population with the smallest θ_k has rank 1, the population with the second smallest θ_k has rank 2, and so on. (Alternatively, the ranks are reversed in Table 1, so the state with the largest estimate has estimated rank 1, and so on.) Formally, we define the rank for the k^{th} population as

$$r_k = \sum_{j=1}^{K} I(\theta_j \leq \theta_k) = 1 + \sum_{j:j\neq k} I(\theta_j \leq \theta_k), \quad \text{for} \quad k = 1, ..., K. \tag{3}$$

The estimated ranking, computed based on the estimates $\hat{\theta}_1, ..., \hat{\theta}_K$, is denoted by $(\hat{r}_1, ..., \hat{r}_K)$, where

$$\hat{r}_k = 1 + \sum_{j:j\neq k} I(\hat{\theta}_j \leq \hat{\theta}_k), \quad \text{for} \quad k = 1, 2, ..., K. \tag{4}$$

Naturally, uncertainty in the estimators $\hat{\theta}_1, ..., \hat{\theta}_K$ is propagated to the estimated ranking. An easily understandable measure of uncertainty should accompany a released ranking.

While the values of $\theta_1, ..., \theta_K$ are unknown, suppose for each $k \in \{1, 2, ..., K\}$ we know real numbers $L_k < U_k$ such that

$$\theta_k \in (L_k, U_k). \tag{5}$$

That is, while each θ_k is unknown, we do know θ_k is contained in the open interval (L_k, U_k). Note that for the purpose of deriving a confidence region for the ranking, there will be no loss of generality in assumption (5) because when we construct the confidence region in Section 3, we will replace the intervals in (5) with joint confidence intervals and the Main Result will then be used to obtain a probability statement.

For each $k \in \{1, 2, ..., K\}$, define

$$\begin{aligned} I_k &= \{1, 2, ..., K\}\backslash\{k\}, \\ \Lambda_{Lk} &= \{j \in I_k : U_j \leq L_k\}, \\ \Lambda_{Rk} &= \{j \in I_k : U_k \leq L_j\}, \\ \Lambda_{Ok} &= \{j \in I_k : U_j > L_k \text{ and } U_k > L_j\} = I_k \backslash (\Lambda_{Lk} \cup \Lambda_{Rk}). \end{aligned} \tag{6}$$

For each $k \in \{1, 2, ..., K\}$, and $j \in I_k$, note that

1. $j \in \Lambda_{Lk}$ if and only if $(L_j, U_j) \cap (L_k, U_k) = \emptyset$ and (L_j, U_j) lies to the left of (L_k, U_k);
2. $j \in \Lambda_{Rk}$ if and only if $(L_j, U_j) \cap (L_k, U_k) = \emptyset$ and (L_j, U_j) lies to the right of (L_k, U_k);
3. $j \in \Lambda_{Ok}$ if and only if $(L_j, U_j) \cap (L_k, U_k) \neq \emptyset$.

It follows that Λ_{Lk}, Λ_{Rk}, and Λ_{Ok} are mutually exclusive, and $\Lambda_{Lk} \cup \Lambda_{Rk} \cup \Lambda_{Ok} = I_k$. For a finite set A, let $|A|$ denote the number of elements in A.

Main Result. Under the scenario described above, it follows that for each $k \in \{1, 2, ..., K\}$,

$$r_k \in \{|\Lambda_{Lk}| + 1, |\Lambda_{Lk}| + 2, |\Lambda_{Lk}| + 3, ..., |\Lambda_{Lk}| + |\Lambda_{Ok}| + 1\}. \tag{7}$$

Proof: Let $k \in \{1, 2, ..., K\}$. Because Λ_{Lk}, Λ_{Rk}, and Λ_{Ok} are mutually exclusive, and $\Lambda_{Lk} \cup \Lambda_{Rk} \cup \Lambda_{Ok} = I_k$, we can write the rank of the k^{th} population as follows:

$$\begin{aligned} r_k = 1 + \sum_{j:j\neq k} I(\theta_j \leq \theta_k) &= 1 + \sum_{j \in I_k} I(\theta_j \leq \theta_k) \\ &= 1 + \sum_{j \in \Lambda_{Lk}} I(\theta_j \leq \theta_k) + \sum_{j \in \Lambda_{Rk}} I(\theta_j \leq \theta_k) + \sum_{j \in \Lambda_{Ok}} I(\theta_j \leq \theta_k). \end{aligned} \tag{8}$$

We note that $j \in \Lambda_{Lk} \Longrightarrow U_j \le L_k \Longrightarrow L_j < \theta_j < U_j \le L_k < \theta_k < U_k \Longrightarrow I(\theta_j \le \theta_k) = 1$; and $j \in \Lambda_{Rk} \Longrightarrow U_k \le L_j \Longrightarrow L_k < \theta_k < U_k \le L_j < \theta_j < U_j \Longrightarrow I(\theta_j \le \theta_k) = 0$; and therefore, continuing from equation (8), we have:

$$\begin{aligned} r_k &= 1 + \sum_{j \in \Lambda_{Lk}} I(\theta_j \le \theta_k) + \sum_{j \in \Lambda_{Rk}} I(\theta_j \le \theta_k) + \sum_{j \in \Lambda_{Ok}} I(\theta_j \le \theta_k) \\ &= 1 + \sum_{j \in \Lambda_{Lk}} 1 + \sum_{j \in \Lambda_{Rk}} 0 + \sum_{j \in \Lambda_{Ok}} I(\theta_j \le \theta_k) \\ &= 1 + |\Lambda_{Lk}| + \sum_{j \in \Lambda_{Ok}} I(\theta_j \le \theta_k). \end{aligned}$$

Because $\sum_{j \in \Lambda_{Ok}} I(\theta_j \le \theta_k) \in \{0, 1, ..., |\Lambda_{Ok}|\}$ it follows that

$$r_k = 1 + |\Lambda_{Lk}| + \sum_{j \in \Lambda_{Ok}} I(\theta_j \le \theta_k) \in \{|\Lambda_{Lk}| + 1, |\Lambda_{Lk}| + 2, |\Lambda_{Lk}| + 3, ..., |\Lambda_{Lk}| + |\Lambda_{Ok}| + 1\}.$$

This completes the proof. $\square$

3. JOINT CONFIDENCE REGION FOR A RANKING

Assume that $\{(L_1, U_1), (L_2, U_2), ..., (L_K, U_K)\}$ is a collection of confidence intervals for the unknown parameters θ_1, θ_2, ..., θ_K, respectively, and the joint coverage probability of these intervals is greater than or equal to $1 - \alpha$. That is, we assume that

$$P\Big[\bigcap_{k=1}^{K} \Big\{\theta_k \in (L_k, U_k)\Big\}\Big] \ge 1 - \alpha.$$

In this setting, L_1, L_2, ..., L_K, U_1, U_2, ..., U_K are random variables. By the Main Result,

$$\bigcap_{k=1}^{K} \Big\{\theta_k \in (L_k, U_k)\Big\} \Longrightarrow \bigcap_{k=1}^{K} \Big\{r_k \in \{|\Lambda_{Lk}| + 1, |\Lambda_{Lk}| + 2, |\Lambda_{Lk}| + 3, ..., |\Lambda_{Lk}| + |\Lambda_{Ok}| + 1\}\Big\},$$

where for each $k \in \{1, 2, ..., K\}$, r_k is the rank defined in (3), and Λ_{Lk} and Λ_{Ok} are as defined in (6). Therefore, it follows that:

$$\begin{aligned} P\Big[\bigcap_{k=1}^{K} \Big\{r_k \in \{|\Lambda_{Lk}| + 1, |\Lambda_{Lk}| + 2, |\Lambda_{Lk}| + 3, ..., |\Lambda_{Lk}| + |\Lambda_{Ok}| + 1\}\Big\}\Big] &\ge \\ P\Big[\bigcap_{k=1}^{K} \Big\{\theta_k \in (L_k, U_k)\Big\}\Big] &\ge 1 - \alpha. \end{aligned}$$

Thus we have shown that

$$\Big\{(r_1, ..., r_K) : r_k \in \{|\Lambda_{Lk}| + 1, |\Lambda_{Lk}| + 2, |\Lambda_{Lk}| + 3, ..., |\Lambda_{Lk}| + |\Lambda_{Ok}| + 1\} \text{ for } k = 1, ..., K\Big\} \quad (9)$$

is a *joint confidence region* (or set) for the ranking $(r_1, ..., r_K)$ having joint coverage probability of at least $1 - \alpha$.

The following result shows that if the estimator $\hat{\theta}_k \in (L_k, U_k)$ for all $k \in \{1, 2, ..., K\}$ with probability 1, then the estimated ranking $(\hat{r}_1, \hat{r}_2, ..., \hat{r}_K)$ is contained in the joint confidence region (9) with probability 1.

Result 3.1. If $P\left[\bigcap_{k=1}^{K}\left\{\hat{\theta}_k \in (L_k, U_k)\right\}\right] = 1$, then

$$P\left[\bigcap_{k=1}^{K}\left\{\hat{r}_k \in \{|\Lambda_{Lk}|+1, |\Lambda_{Lk}|+2, |\Lambda_{Lk}|+3, ..., |\Lambda_{Lk}|+|\Lambda_{Ok}|+1\}\right\}\right] = 1.$$

Proof: If the observed values of $L_1, ..., L_K, U_1, ..., U_K, \hat{\theta}_1, ..., \hat{\theta}_K$ are such that $\hat{\theta}_k \in (L_k, U_k)$ for all $k \in \{1, ..., K\}$, then an argument similar to the one used in the proof of the Main Result gives

$$\begin{aligned} \hat{r}_k &= 1 + \sum_{j:j\neq k} I(\hat{\theta}_j \leq \hat{\theta}_k) \\ &= 1 + |\Lambda_{Lk}| + \sum_{j\in\Lambda_{Ok}} I(\hat{\theta}_j \leq \hat{\theta}_k) \\ &\in \left\{|\Lambda_{Lk}|+1, |\Lambda_{Lk}|+2, |\Lambda_{Lk}|+3, ..., |\Lambda_{Lk}|+|\Lambda_{Ok}|+1\right\}. \end{aligned}$$

for all $k \in \{1, ..., K\}$. Thus we have established that

$$\bigcap_{k=1}^{K}\left\{\hat{\theta}_k \in (L_k, U_k)\right\} \Longrightarrow \bigcap_{k=1}^{K}\left\{\hat{r}_k \in \{|\Lambda_{Lk}|+1, |\Lambda_{Lk}|+2, |\Lambda_{Lk}|+3, ..., |\Lambda_{Lk}|+|\Lambda_{Ok}|+1\}\right\},$$

and therefore,

$$\begin{aligned} P\left[\bigcap_{k=1}^{K}\left\{\hat{r}_k \in \{|\Lambda_{Lk}|+1, |\Lambda_{Lk}|+2, |\Lambda_{Lk}|+3, ..., |\Lambda_{Lk}|+|\Lambda_{Ok}|+1\}\right\}\right] &\geq \\ P\left[\bigcap_{k=1}^{K}\left\{\hat{\theta}_k \in (L_k, U_k)\right\}\right] &= 1. \end{aligned}$$

Hence the result follows. □

In general, the joint confidence region in (9) has more than one ranking (e.g., see Table 3). The following result gives conditions under which the joint confidence region will have only one ranking, and it is $(\hat{r}_1, ..., \hat{r}_K)$. This is the "tightest" possible joint confidence region.

Result 3.2. If $(L_k, U_k) \cap (L_{k'}, U_{k'}) = \emptyset$ for all $k \neq k'$ and $k = 1, 2, ..., K$, then the joint confidence region has only one ranking, and it is the estimated ranking $(\hat{r}_1, ..., \hat{r}_K)$.

Proof: If $(L_k, U_k) \cap (L_{k'}, U_{k'}) = \emptyset$ for all $k \neq k'$, then $|\Lambda_{Ok}| = 0$ for all k. Thus by (9) and Result 3.1, the result follows immediately. □

4. EXAMPLES

Assume $\hat{\theta}_1$, $\hat{\theta}_2$,..., $\hat{\theta}_K$ are independently distributed such that $\hat{\theta}_k \sim N(\theta_k, SE_k)$ for $k = 1, 2, ..., K$ with θ_1, θ_2,..., θ_K unknown and SE_1, SE_2,..., SE_K known. For a given θ_k, an individual $100(1-\alpha)\%$ confidence interval is

$$\Big(\hat{\theta}_k - z_{\frac{\alpha}{2}} SE_k, \hat{\theta}_k + z_{\frac{\alpha}{2}} SE_k\Big).$$

We consider two cases of the joint confidence intervals for $\theta_1, ..., \theta_K$: (1) using Bonferroni correction; and (2) using independence.

4.1 Joint Confidence Intervals for $\theta_1, ..., \theta_K$ Using Bonferroni Correction

We apply the Bonferroni correction to get a collection of confidence intervals whose joint coverage for θ_1, θ_2, ..., θ_K is greater than or equal to $1-\alpha$; these intervals are given by

$$\Big(\hat{\theta}_k - z_{\frac{(\alpha/K)}{2}} SE_k, \hat{\theta}_k + z_{\frac{(\alpha/K)}{2}} SE_k\Big), \text{ for } k = 1, 2, ..., K. \tag{10}$$

The Bonferroni Inequality states that for events A_1, A_2,..., A_K, we have (Mukhopadhyay 2000, p. 157)

$$P\Big\{\bigcap_{i=1}^{K} A_i\Big\} \geq \sum_{i=1}^{K} P(A_i) - (K-1).$$

Applying the Bonferroni Inequality, we see that

$$\begin{aligned} P\Big\{\bigcap_{k=1}^{K}\Big\{\theta_k \in \Big(\hat{\theta}_k - z_{\frac{(\alpha/K)}{2}} SE_k, \hat{\theta}_k + z_{\frac{(\alpha/K)}{2}} SE_k\Big)\Big\}\Big\} \\ \geq \sum_{k=1}^{K} P\Big\{\theta_k \in \Big(\hat{\theta}_k - z_{\frac{(\alpha/K)}{2}} SE_k, \hat{\theta}_k + z_{\frac{(\alpha/K)}{2}} SE_k\Big)\Big\} - (K-1) \\ = \sum_{k=1}^{K}(1-\frac{\alpha}{K}) - (K-1) = K - \alpha - (K-1) = 1-\alpha. \end{aligned}$$

Thus the Bonferroni corrected confidence intervals given by (10) have joint coverage probability greater than or equal to $1-\alpha$. We apply the proposed methodology to the American Community Survey travel time to work data for the year 2011. In this example, θ_k is the mean travel time (in minutes) to work for state k (including Washington, D.C.) where $k = 1, 2, ..., 51$. Table 2 shows the Bonferroni corrected joint confidence intervals for θ_1, θ_2,..., θ_{51} as given by (10) with $\alpha = 0.10$. This table also shows the joint confidence region for the ranking $(r_1, ..., r_{51})$ obtained by using (9) as applied to the Bonferroni corrected confidence intervals for θ_1, θ_2,..., θ_K.

To illustrate the details of one row of Table 2, we focus on Illinois. For $\alpha = 0.10$, $z_{\frac{(\alpha/51)}{2}} = 3.1$. The Bonferroni corrected joint confidence interval for $\theta_{Illinois}$ is given by (10)

$$\Big(28.17 - 3.1(0.11), 28.17 + 3.1(0.11)\Big) = \Big(27.8294, 28.5106\Big). \tag{11}$$

To obtain the portion of the joint confidence region for $r_{Illinois}$, we refer to the observed ranking and note that

$$\begin{aligned}
\Lambda_{L,Illinois} &= \{\text{ California, Georgia, ..., South Dakota }\} \text{ implies } |\Lambda_{L,Illinois}| = 44; \\
\Lambda_{R,Illinois} &= \{\text{ Maryland, New York, New Jersey, District of Columbia }\} \text{ implies} \\
&\quad |\Lambda_{R,Illinois}| = 4; \text{ and} \\
\Lambda_{O,Illinois} &= \{\text{ Massachusetts, Virginia }\} \text{ implies } |\Lambda_{O,Illinois}| = 2.
\end{aligned}$$

Hence the portion of the joint confidence region for $r_{Illinois}$ using (9) is

$$\{44+1, 44+2, 44+2+1\} = \{45, 46, 47\}. \tag{12}$$

The other rows of Table 2 are obtained similarly.

For each rank r_k from 1 to 51, Figure 1 shows which states can occupy that rank, e.g., District of Columbia or New Jersey can occupy $r_k = 48$. We assume no ties. Figure 1 makes it easy to identify all overall rankings in the 90% joint confidence region as specified in (9).

Five of the many rankings in the 90% joint confidence region are given in the columns of Table 3. Note that Ranking 1 is the observed estimated ranking and that it is highlighted in the 90% joint confidence region of Figure 1 (See Result 3.1).

As noted earlier, each row of the joint confidence region (Figure 1) shows which states could occupy each rank. Similarly, each column k of the joint confidence region (Figure 1) shows the marginal confidence set for the rank r_k of state k.

4.2 Joint Confidence Intervals for $\theta_1, ..., \theta_K$ Using Independence

In this situation, because $\hat{\theta}_1$,..., $\hat{\theta}_K$ are independently distributed such that $\hat{\theta}_k \sim N(\theta_k, SE_k)$ for $k = 1, 2, ..., K$ with θ_1, θ_2, ..., θ_K unknown and SE_1, SE_2, ..., SE_K known, we may also consider the following intervals whose joint coverage equals $1-\alpha$:

$$\left(\hat{\theta}_k - z_{\frac{\gamma}{2}} SE_k, \hat{\theta}_k + z_{\frac{\gamma}{2}} SE_k\right), \text{ for } k = 1, 2, ..., K, \tag{13}$$

where $\gamma = 1-(1-\alpha)^{1/K}$. We note that

$$\begin{aligned}
&P\left\{\bigcap_{k=1}^{K}\{\theta_k \in \left(\hat{\theta}_k - z_{\frac{\gamma}{2}} SE_k, \hat{\theta}_k + z_{\frac{\gamma}{2}} SE_k\right)\}\right\} \\
&= P\left\{-z_{\frac{\gamma}{2}} < \frac{\hat{\theta}_1 - \theta_1}{SE_1} < z_{\frac{\gamma}{2}}\ , -z_{\frac{\gamma}{2}} < \frac{\hat{\theta}_2 - \theta_2}{SE_2} < z_{\frac{\gamma}{2}}\ , ..., -z_{\frac{\gamma}{2}} < \frac{\hat{\theta}_K - \theta_K}{SE_K} < z_{\frac{\gamma}{2}}\right\}
\end{aligned}$$

$$= \prod_{k=1}^{K} P\Big\{ -z_{\frac{\gamma}{2}} < \frac{\hat{\theta}_k - \theta_k}{SE_k} < z_{\frac{\gamma}{2}} \Big\}$$

$$= \prod_{k=1}^{K} [1-\gamma] = [1-\gamma]^K = [1-(1-(1-\alpha)^{1/K})]^K = (1-\alpha).$$

Table 2: Travel Time To Work Data, Using Bonferroni Joint Confidence Intervals (10) for $\theta_1, ..., \theta_K$

$\hat{r}_k$	State	$\hat{\theta}_k$	SE_k	90% Joint Confidence Intervals for θ_k's	90% Joint Confidence Region for r_k's
51	Maryland	32.21	0.15	$(31.7456, 32.6744)$	$\{50, 51\}$
50	New York	31.50	0.09	$(31.2214, 31.7786)$	$\{50, 51\}$
49	New Jersey	30.53	0.12	$(30.1585, 30.9015)$	$\{48, 49\}$
48	District of Columbia	30.10	0.32	$(29.1092, 31.0908)$	$\{48, 49\}$
47	Illinois	28.17	0.11	$(27.8294, 28.5106)$	$\{45, 46, 47\}$
46	Massachusetts	27.99	0.13	$(27.5875, 28.3925)$	$\{43, 44, ..., 47\}$
45	Virginia	27.74	0.13	$(27.3375, 28.1425)$	$\{42, 43, ..., 47\}$
44	California	27.14	0.07	$(26.9233, 27.3567)$	$\{42, 43, 44, 45\}$
43	Georgia	27.11	0.17	$(26.5837, 27.6363)$	$\{42, 43, ..., 46\}$
42	New Hampshire	26.90	0.30	$(25.9712, 27.8288)$	$\{37, 38, ..., 46\}$
41	Pennsylvania	25.92	0.09	$(25.6414, 26.1986)$	$\{36, 37, ..., 42\}$
40	Florida	25.76	0.11	$(25.4194, 26.1006)$	$\{35, 36, ..., 42\}$
39	Hawaii	25.69	0.27	$(24.8541, 26.5259)$	$\{30, 31, ..., 42\}$
38	West Virginia	25.58	0.31	$(24.6202, 26.5398)$	$\{29, 30, ..., 42\}$
37	Washington	25.51	0.14	$(25.0765, 25.9435)$	$\{33, 34, ..., 41\}$
36	Delaware	25.30	0.37	$(24.1544, 26.4456)$	$\{24, 25, ..., 42\}$
35	Connecticut	24.98	0.19	$(24.3917, 25.5683)$	$\{27, 28, ..., 40\}$
34	Texas	24.82	0.07	$(24.6033, 25.0367)$	$\{29, 30, ..., 38\}$
33	Arizona	24.76	0.15	$(24.2956, 25.2244)$	$\{26, 27, ..., 39\}$
32	Louisiana	24.54	0.15	$(24.0756, 25.0044)$	$\{23, 24, ..., 38\}$
31	Colorado	24.51	0.19	$(23.9217, 25.0983)$	$\{23, 24, ..., 39\}$
30	Tennessee	24.23	0.14	$(23.7965, 24.6635)$	$\{23, 24, ..., 37\}$
29	Michigan	24.11	0.10	$(23.8004, 24.4196)$	$\{23, 24, ..., 35\}$
28	Nevada	24.10	0.27	$(23.2641, 24.9359)$	$\{17, 18, ..., 38\}$
27	Alabama	23.94	0.14	$(23.5065, 24.3735)$	$\{21, 22, ..., 34\}$
26	Mississippi	23.86	0.24	$(23.1169, 24.6031)$	$\{17, 18, ..., 35\}$
25	South Carolina	23.61	0.16	$(23.1146, 24.1054)$	$\{17, 18, ..., 32\}$
24	Indiana	23.45	0.11	$(23.1094, 23.7906)$	$\{17, 18, ..., 28\}$
23	Maine	23.41	0.25	$(22.6360, 24.1840)$	$\{15, 16, ..., 33\}$
22	North Carolina	23.37	0.12	$(22.9985, 23.7415)$	$\{16, 17, ..., 28\}$
21	Rhode Island	23.36	0.29	$(22.4621, 24.2579)$	$\{15, 16, ..., 33\}$
20	Ohio	23.12	0.09	$(22.8414, 23.3986)$	$\{15, 16, ..., 27\}$
19	Missouri	23.07	0.13	$(22.6675, 23.4725)$	$\{15, 16, ..., 27\}$
18	Minnesota	22.99	0.10	$(22.6804, 23.2996)$	$\{15, 16, ..., 27\}$
17	Kentucky	22.86	0.15	$(22.3956, 23.3244)$	$\{15, 16, ..., 27\}$
16	Oregon	22.54	0.16	$(22.0446, 23.0354)$	$\{12, 13, ..., 23\}$
15	Vermont	21.94	0.31	$(20.9802, 22.8998)$	$\{10, 11, ..., 22\}$
14	Wisconsin	21.92	0.11	$(21.5794, 22.2606)$	$\{10, 11, ..., 16\}$
13	Utah	21.61	0.20	$(20.9908, 22.2292)$	$\{10, 11, ..., 16\}$
12	New Mexico	21.43	0.27	$(20.5941, 22.2659)$	$\{10, 11, ..., 16\}$
11	Arkansas	21.31	0.23	$(20.5979, 22.0221)$	$\{10, 11, ..., 15\}$
10	Oklahoma	21.13	0.15	$(20.6656, 21.5944)$	$\{10, 11, ..., 15\}$
9	Idaho	19.66	0.24	$(18.9169, 20.4031)$	$\{4, 5, ..., 9\}$
8	Kansas	18.90	0.16	$(18.4046, 19.3954)$	$\{3, 4, ..., 9\}$
7	Iowa	18.77	0.13	$(18.3675, 19.1725)$	$\{3, 4, ..., 9\}$
6	Alaska	18.39	0.33	$(17.3683, 19.4117)$	$\{1, 2, ..., 9\}$
5	Montana	18.18	0.32	$(17.1892, 19.1708)$	$\{1, 2, ..., 9\}$
4	Wyoming	18.10	0.50	$(16.5519, 19.6481)$	$\{1, 2, ..., 9\}$
3	Nebraska	18.06	0.19	$(17.4717, 18.6483)$	$\{1, 2, ..., 8\}$
2	North Dakota	16.91	0.36	$(15.7954, 18.0246)$	$\{1, 2, ..., 6\}$
1	South Dakota	16.86	0.28	$(15.9931, 17.7269)$	$\{1, 2, ..., 6\}$

Source: Based on 2011 1-Year American Community Survey, Ranking Table R0801.

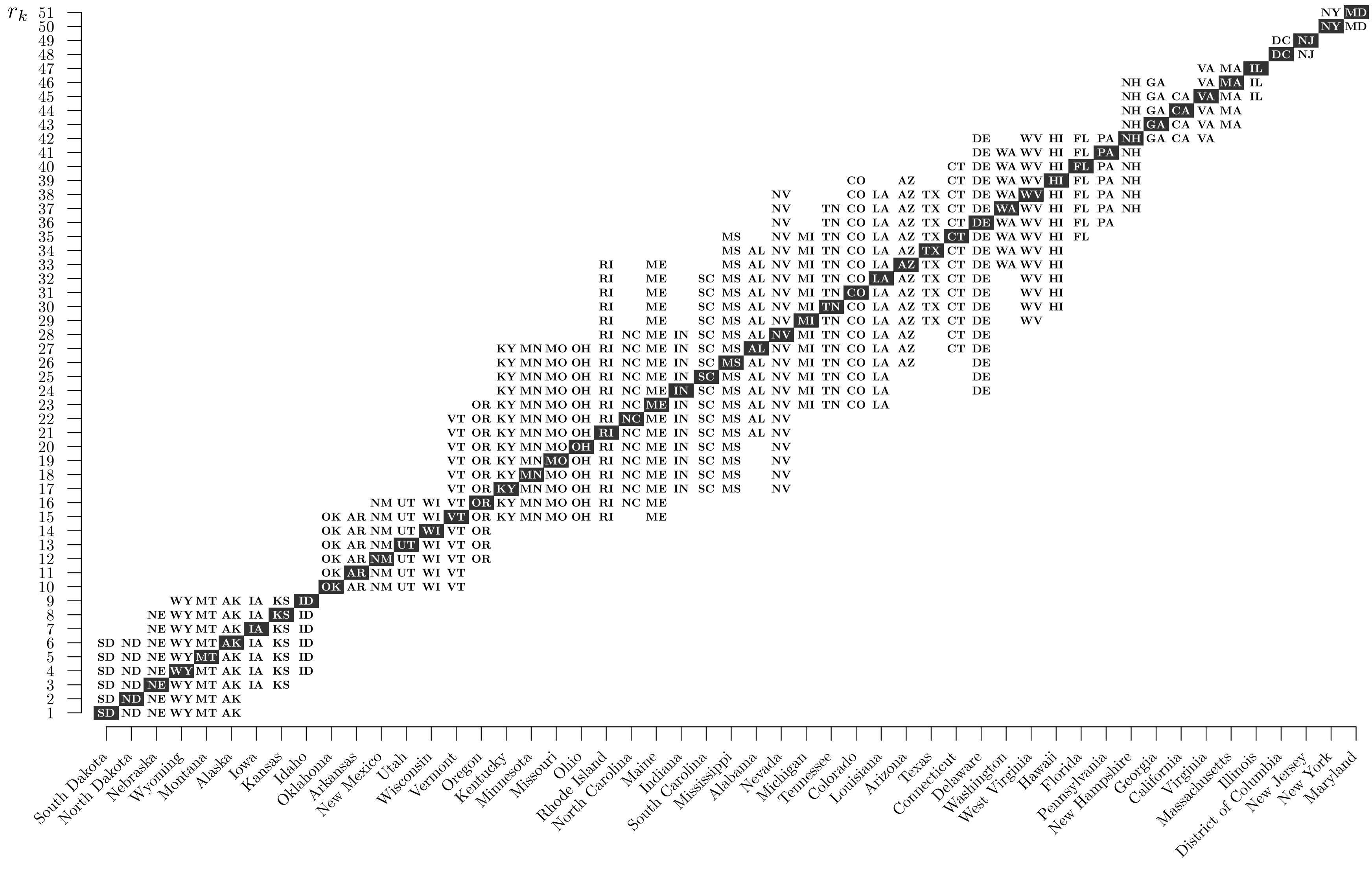

Figure 1: Visualization of the 90% Joint Confidence Region for the Ranking Using Travel Time to Work Data. Highlighted States in Joint Confidence Region Show the Observed Estimated Ranking 1 of Table 3. Each Specific Row of the Joint Confidence Region Shows Which States Could Occupy the Associated Rank; Each Specific Column of the Joint Confidence Region Shows Which Ranks the Associated State Could Occupy. For example, Rank (Row) 10 Could Be Occupied by States OK, AR, NM, UT, WI, or VT; State (Column) Minnesota Could Occupy Ranks 15, 16, 17, 18, 19,..., 26, or 27. *Source*: Based on Data from 2011 1-Year American Community Survey, Ranking Table R0801.

Table 3: Five of Many Rankings in the 90% Joint Confidence Region of Figure 1

Rank r_k	**Ranking 1**	**Ranking 2**	**Ranking 3**	**Ranking 4**	**Ranking 5**
51	Maryland	New York	Maryland	New York	Maryland
50	New York	Maryland	New York	Maryland	New York
49	New Jersey	District of Columbia	New Jersey	District of Columbia	District of Columbia
48	District of Columbia	New Jersey	District of Columbia	New Jersey	New Jersey
47	Illinois	Illinois	Illinois	Virginia	Illinois
46	Massachusetts	Massachusetts	Massachusetts	Illinois	Massachusetts
45	Virginia	Virginia	Virginia	Massachusetts	Virginia
44	California	California	California	Georgia	California
43	Georgia	Georgia	Georgia	California	Georgia
42	New Hampshire	New Hampshire	New Hampshire	New Hampshire	New Hampshire
41	Pennsylvania	Pennsylvania	Pennsylvania	Pennsylvania	Pennsylvania
40	Florida	Florida	Florida	Florida	Florida
39	Hawaii	Hawaii	Hawaii	Hawaii	Hawaii
38	West Virginia	West Virginia	West Virginia	West Virginia	West Virginia
37	Washington	Washington	Washington	Washington	Washington
36	Delaware	Delaware	Delaware	Delaware	Delaware
35	Connecticut	Connecticut	Connecticut	Connecticut	Connecticut
34	Texas	Texas	Texas	Texas	Texas
33	Arizona	Arizona	Arizona	Arizona	Arizona
32	Louisiana	Louisiana	Louisiana	Colorado	Louisiana
31	Colorado	Colorado	Tennessee	Louisiana	Colorado
30	Tennessee	Tennessee	Michigan	Michigan	Tennessee
29	Michigan	Michigan	Colorado	Tennessee	Michigan
28	Nevada	Nevada	Nevada	Alabama	Nevada
27	Alabama	Alabama	Alabama	Nevada	Alabama
26	Mississippi	Mississippi	Mississippi	Mississippi	Mississippi
25	South Carolina	South Carolina	South Carolina	South Carolina	South carolina
24	Indiana	Indiana	Indiana	Indiana	Indiana
23	Maine	Maine	Maine	Maine	Maine
22	North Carolina	North Carolina	North Carolina	Rhode Island	North Carolina
21	Rhode Island	Rhode Island	Rhode Island	North Carolina	Rhode Island
20	Ohio	Ohio	Ohio	Missouri	Ohio
19	Missouri	Missouri	Kentucky	Ohio	Missouri
18	Minnesota	Minnesota	Minnesota	Minnesota	Minnesota
17	Kentucky	Kentucky	Missouri	Kentucky	Kentucky
16	Oregon	Oregon	Oregon	Oregon	Oregon
15	Vermont	Vermont	Vermont	Vermont	Vermont
14	Wisconsin	Wisconsin	Wisconsin	Wisconsin	Wisconsin
13	Utah	Utah	Utah	Utah	Utah
12	New Mexico	New Mexico	New Mexico	New Mexico	New Mexico
11	Arkansas	Arkansas	Arkansas	Arkansas	Arkansas
10	Oklahoma	Oklahoma	Oklahoma	Oklahoma	Oklahoma
9	Idaho	Idaho	Idaho	Iowa	Idaho
8	Kansas	Kansas	Kansas	Idaho	Kansas
7	Iowa	Iowa	Iowa	Kansas	Iowa
6	Alaska	Alaska	South Dakota	Alaska	Alaska
5	Montana	Montana	Alaska	Montana	Montana
4	Wyoming	Nebraska	Montana	Wyoming	Wyoming
3	Nebraska	Wyoming	Wyoming	Nebraska	Nebraska
2	North Dakota	South Dakota	Nebraska	North Dakota	North Dakota
1	South Dakota	North Dakota	North Dakota	South Dakota	South Dakota

Source: Based on Data from 2011 1-Year American Community Survey, Ranking Table R0801.

Thus the confidence intervals given by (13) have joint coverage probability equal to $1-\alpha$. As with the Bonferroni corrected confidence intervals, we apply this proposed methodology to the American Community Survey travel time to work data. Table 4 shows the joint confidence intervals for $\theta_1, \theta_2, ..., \theta_{51}$ as given by (13) with $\alpha = 0.10$. This table also shows the joint confidence region for the ranking $(r_1, r_2, ..., r_{51})$ obtained by using (9) as applied to the independent confidence intervals for $\theta_1, \theta_2, ..., \theta_{51}$.

From Tables 2 and 4, note that the confidence intervals for θ_k are shorter in Table 4 than in Table 2; and this will always be the case as shown in Result 4.1. More precisely, the confidence intervals for θ_k in Table 4 will never be longer than the corresponding confidence intervals in Table 2. As a consequence, the joint confidence region for $(r_1, ..., r_{51})$ based on

Table 4: Travel Time To Work Data, Using Independent Joint Confidence Intervals (13) for $\theta_1, ..., \theta_K$

$\hat{r}_k$	State	$\hat{\theta}_k$	SE_k	90% Joint Confidence Intervals for θ_k's	90% Joint Confidence Region for r_k's
51	Maryland	32.21	0.15	$(31.7479, 32.6721)$	$\{50, 51\}$
50	New York	31.50	0.09	$(31.2227, 31.7773)$	$\{50, 51\}$
49	New Jersey	30.53	0.12	$(30.1603, 30.8997)$	$\{48, 49\}$
48	District of Columbia	30.10	0.32	$(29.1141, 31.0895)$	$\{48, 49\}$
47	Illinois	28.17	0.11	$(27.8311, 28.5089)$	$\{45, 46, 47\}$
46	Massachusetts	27.99	0.13	$(27.5895, 28.3905)$	$\{43, 44, ..., 47\}$
45	Virginia	27.74	0.13	$(27.3395, 28.1405)$	$\{42, 43, ..., 47\}$
44	California	27.14	0.07	$(26.9243, 27.3557)$	$\{42, 43, 44, 45\}$
43	Georgia	27.11	0.17	$(26.5862, 27.6338)$	$\{42, 43, ..., 46\}$
42	New Hampshire	26.90	0.30	$(25.9757, 27.8243)$	$\{37, 38, ..., 46\}$
41	Pennsylvania	25.92	0.09	$(25.6427, 26.1973)$	$\{36, 37, ..., 42\}$
40	Florida	25.76	0.11	$(25.4211, 26.0989)$	$\{35, 36, ..., 42\}$
39	Hawaii	25.69	0.27	$(24.8582, 26.5218)$	$\{30, 31, ..., 42\}$
38	West Virginia	25.58	0.31	$(24.6249, 26.5351)$	$\{29, 30, ..., 42\}$
37	Washington	25.51	0.14	$(25.0787, 25.9413)$	$\{33, 34, ..., 41\}$
36	Delaware	25.30	0.37	$(24.1601, 26.4399)$	$\{24, 25, ..., 42\}$
35	Connecticut	24.98	0.19	$(24.3946, 25.5654)$	$\{27, 28, ..., 40\}$
34	Texas	24.82	0.07	$(24.6043, 25.0357)$	$\{29, 30, ..., 38\}$
33	Arizona	24.76	0.15	$(24.2979, 25.2221)$	$\{26, 27, ..., 39\}$
32	Louisiana	24.54	0.15	$(24.0779, 25.0021)$	$\{23, 24, ..., 38\}$
31	Colorado	24.51	0.19	$(23.9246, 25.0954)$	$\{23, 24, ..., 39\}$
30	Tennessee	24.23	0.14	$(23.7987, 24.6613)$	$\{23, 24, ..., 37\}$
29	Michigan	24.11	0.10	$(23.8019, 24.4181)$	$\{23, 24, ..., 35\}$
28	Nevada	24.10	0.27	$(23.2682, 24.9318)$	$\{17, 18, ..., 38\}$
27	Alabama	23.94	0.14	$(23.5087, 24.3713)$	$\{21, 22, ..., 34\}$
26	Mississippi	23.86	0.24	$(23.1206, 24.5994)$	$\{17, 18, ..., 35\}$
25	South Carolina	23.61	0.16	$(23.1171, 24.1029)$	$\{17, 18, ..., 32\}$
24	Indiana	23.45	0.11	$(23.1111, 23.7889)$	$\{17, 18, ..., 28\}$
23	Maine	23.41	0.25	$(22.6398, 24.1802)$	$\{15, 16, ..., 33\}$
22	North Carolina	23.37	0.12	$(23.0003, 23.7397)$	$\{16, 17, ..., 28\}$
21	Rhode Island	23.36	0.29	$(22.4665, 24.2535)$	$\{15, 16, ..., 33\}$
20	Ohio	23.12	0.09	$(22.8427, 23.3973)$	$\{15, 16, ..., 27\}$
19	Missouri	23.07	0.13	$(22.6695, 23.4705)$	$\{15, 16, ..., 27\}$
18	Minnesota	22.99	0.10	$(22.6819, 23.2981)$	$\{15, 16, ..., 27\}$
17	Kentucky	22.86	0.15	$(22.3979, 23.3221)$	$\{15, 16, ..., 27\}$
16	Oregon	22.54	0.16	$(22.0471, 23.0329)$	$\{12, 13, ..., 23\}$
15	Vermont	21.94	0.31	$(20.9849, 22.8951)$	$\{10, 11, ..., 22\}$
14	Wisconsin	21.92	0.11	$(21.5811, 22.2589)$	$\{10, 11, ..., 16\}$
13	Utah	21.61	0.20	$(20.9938, 22.2262)$	$\{10, 11, ..., 16\}$
12	New Mexico	21.43	0.27	$(20.5982, 22.2618)$	$\{10, 11, ..., 16\}$
11	Arkansas	21.31	0.23	$(20.6014, 22.0186)$	$\{10, 11, ..., 15\}$
10	Oklahoma	21.13	0.15	$(20.6679, 21.5921)$	$\{10, 11, ..., 15\}$
9	Idaho	19.66	0.24	$(18.9206, 20.3994)$	$\{4, 5, ..., 9\}$
8	Kansas	18.90	0.16	$(18.4071, 19.3929)$	$\{3, 4, ..., 9\}$
7	Iowa	18.77	0.13	$(18.3695, 19.1705)$	$\{3, 4, ..., 9\}$
6	Alaska	18.39	0.33	$(17.3733, 19.4067)$	$\{1, 2, ..., 9\}$
5	Montana	18.18	0.32	$(17.1941, 19.1659)$	$\{1, 2, ..., 9\}$
4	Wyoming	18.10	0.50	$(16.5596, 19.6404)$	$\{1, 2, ..., 9\}$
3	Nebraska	18.06	0.19	$(17.4746, 18.6454)$	$\{1, 2, ..., 8\}$
2	North Dakota	16.91	0.36	$(15.8009, 18.0191)$	$\{1, 2, ..., 6\}$
1	South Dakota	16.86	0.28	$(15.9973, 17.7227)$	$\{1, 2, ..., 6\}$

Source: Based on 2011 1-Year American Community Survey, Ranking Table R0801.

independence is at least as tight as the corresponding confidence region based on Bonferroni correction. From Tables 2 and 4, the joint confidence regions are the same for this data.

Result 4.1. The intervals in (13) based on independence are shorter than the corresponding intervals in (10) based on Bonferroni correction.

Proof: Note that the intervals in (13) are shorter than the corresponding ones in (10) if and only if $z_{\frac{\gamma}{2}} < z_{\frac{(\alpha/K)}{2}}$, which is equivalent to

$$1 - \alpha < \left(1 - \frac{\alpha}{K}\right)^K. \tag{14}$$

Thus it is sufficient to show that the inequality in (14) is true. By the Binomial Theorem (recall $K \geq 2$),

$$\left(1 - \frac{\alpha}{K}\right)^K = \sum_{j=0}^{K} \binom{K}{j} \left(\frac{-\alpha}{K}\right)^j = 1 - \alpha + \sum_{j=2}^{K} \binom{K}{j} \left(\frac{-\alpha}{K}\right)^j. \tag{15}$$

There are two cases to consider.

Case 1, K Is Odd Positive Integer: The sum $S = \sum_{j=2}^{K} \binom{K}{j} \left(\frac{-\alpha}{K}\right)$ contains an even number of terms. For the binomials in (16), where $j = 2, 4, ..., K-1$, we have

$$\begin{aligned} \binom{K}{j}\left(\frac{-\alpha}{K}\right)^j + \binom{K}{j+1}\left(\frac{-\alpha}{K}\right)^{j+1} &= \binom{K}{j}\left(\frac{\alpha^j}{K^j}\right) - \binom{K}{j+1}\left(\frac{\alpha^{j+1}}{K^{j+1}}\right) \\ &= \frac{K!}{j!(K-j-1)!}\left(\frac{\alpha^j}{K^{j+1}}\right)\left[\frac{K}{K-j} - \frac{\alpha}{j+1}\right]. \end{aligned} \tag{16}$$

Note that S is a sum of binomials of the form given in (16). Because $\frac{K}{K-j} > \frac{\alpha}{j+1}$ for all $j = 2, 4, 6, ..., K-1$, all binomials in (16) are positive. Hence, the sum $S > 0$; and from (15), the result in (14) has been shown when K is an odd positive integer.

Case 2, K Is Even Positive Integer: The sum S contains an odd number of terms, and the last term is $\frac{\alpha^K}{K^K}$, which is positive. All of the remaining terms in S can be written as binomials of the form given in (16), which are all positive. Hence, the sum $S > 0$; and again from (15), the result in (14) has been shown when K is an even positive integer.

Thus Result 4.1 has been shown. $\square$

5. SIMULATION STUDY

The purpose of this simulation study is to compute the actual coverage probability of the joint confidence region (9) when $\hat{\theta}_1$, $\hat{\theta}_2$, ..., $\hat{\theta}_K$ are independently distributed with $\hat{\theta}_k \sim N(\theta_k, SE_k)$, $k = 1, 2, ..., K$ and the joint $100(1-\alpha)\%$ confidence intervals for θ_1, θ_2, ..., θ_K are computed using the Bonferroni corrected intervals given by (10).

We set θ_1, θ_2, ..., θ_K equal to the estimates given in Table 2; $r_1, r_2, ..., r_K$ are set equal to the estimates in Table 2; SE_1, SE_2,..., SE_K are set equal to the values given in Table 2; and hence $K = 51$. The simulation proceeds as follows.

1. Draw $\hat{\theta}_k \sim N(\theta_k, SE_k)$, independently for $k = 1, 2, ..., K$.
2. Compute the Bonferroni corrected joint confidence intervals (10) for θ_1, θ_2,..., θ_K with $\alpha = 0.10$.
3. Using the Bonferroni corrected confidence intervals computed in Step 2, use (9) to obtain the joint confidence region for the ranking (r_1,..., r_K).
4. Let $A = I\left[\bigcap_{k=1}^{K}\left\{\theta_k \in \left(\hat{\theta}_k - z_{\frac{(\alpha/K)}{2}} SE_k, \hat{\theta}_k + z_{\frac{(\alpha/K)}{2}} SE_k\right)\right\}\right]$
and $B = I\left[\bigcap_{k=1}^{K}\left\{r_k \in \{|\Lambda_{Lk}|+1, |\Lambda_{Lk}|+2, |\Lambda_{Lk}|+3, ..., |\Lambda_{Lk}|+|\Lambda_{Ok}|+1\}\right\}\right]$.
5. Repeat Steps 1-4 a total of M times to get A_1, A_2,..., A_M and B_1, B_2,..., B_M. A Monte Carlo estimate of the joint coverage probability of the intervals (10) for θ_1, θ_2,..., θ_M is $(1/M)\sum_{i=1}^{M} A_i$, and a Monte Carlo estimate of the joint coverage probability of the region in (9) is $(1/M)\sum_{i=1}^{M} B_i$.

Running the above simulation with $M = 10^5$, we find that the joint coverage probability of (10) and (9) in this setting are 0.904733 and 0.999895, respectively.

Running a similar simulation, we find that the joint coverage probability of (13) and the corresponding region (9) in this setting are 0.899865 and 0.999887, respectively.

6. CONCLUDING REMARKS

A simple and useful $100(1-\alpha)\%$ joint confidence region is given for a ranking $(r_1, r_2, ..., r_K)$ of K populations that gives a measure of uncertainty for the estimated ranking $(\hat{r}_1, \hat{r}_2, ..., \hat{r}_K)$

based on sample survey data. When all confidence intervals for the θ_k do not overlap, the joint confidence region is as "tight" as it can be and only contains the ranking $(\hat{r}_1, ..., \hat{r}_K)$. National statistical agencies may increase the release of rankings now that a measure of uncertainty exists that can be shared with users.

A proposed visualization makes it easy to communicate this uncertainty in the estimated ranking while also revealing many other possible rankings (see some of them in Table 3).

The 90% Joint Confidence Regions are the same in Table 2 (Bonferroni) and Table 4 (independence) because the values of $\frac{\alpha/K}{2} = 0.00098$ and $\frac{\gamma}{2} = 0.00103$ are nearly equal, with corresponding z values 3.096 and 3.081, respectively. Thus the corresponding confidence intervals in (10) and (13) are close as shown in Tables 2 and 4. By Result 4.1, it is possible that the Joint Confidence Regions from the two different approaches could differ, though rarely.

The estimates of mean travel time to work in Table 1 have less precision (one decimal place) than the estimates of mean travel time to work in all remaining tables (two decimal places). The less precision in Table 1 results in some estimates that are equal and hence ties among the estimated ranks. We comment briefly on ties. For simplicity, in the example in Section 4 there are no ties among the point estimates $\hat{\theta}_1, \ldots, \hat{\theta}_K$, nor are there ties among endpoints of the collection of joint confidence intervals $\{(L_1, U_1), \ldots, (L_K, U_K)\}$. Furthermore, we have assumed no ties among the unknown parameters $\theta_1, \ldots, \theta_K$ so that the true unknown population ranking $(r_1, \ldots, r_K)$ is a permutation of the integers $1, \ldots, K$. We believe that our presented theory continues to hold even when there are ties. We note that if the unknown population parameters $\theta_1, \ldots, \theta_K$ are possibly not all distinct, then the true ranking as defined by equation (3) is not necessarily a permutation of the integers $1, \ldots, K$. That is, from equation (3), if there is $k \neq k'$ such that $\theta_k = \theta_{k'}$, then $r_k = r_{k'}$. Therefore if $\theta_1, \ldots, \theta_K$ are possibly not all distinct, then in addition to rankings that are permutations of $1, \ldots, K$, other possible rankings, representing ties among some parameters, may also lie within the joint confidence region in equation (9). In future research, we will further study the issue of ties.

Disclaimer: The views expressed are those of the authors and not necessarily those of the U. S. Bureau of the Census.

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