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Bayesian Hierarchical Spatial Models for Small Area Estimation

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Abstract

For over forty years, the Fay-Herriot model has been extensively used by National Statistical Offices around the world to produce reliable small area statistics. This model develops prediction of small area means of a continuous outcome of interest based on a linear regression on suitable auxiliary variables. Model errors, also known as small area effects, of the Fay-Herriot model are treated as independent and normally distributed zero-mean random variables with an unknown variance. Often population means of geographically contiguous small areas display a spatial pattern. The independence assumption for the random effects may not hold when effective auxiliary variables are unavailable. Lack of suitable covariates to account for the variation of the geographic domain means results in a spatial pattern among the random effects. We consider several spatial random-effects models, including the popular conditional autoregressive and simultaneous autoregressive models as alternatives to the Fay-Herriot model. We carry out a Bayesian analysis of these models based on a class of popular noninformative improper prior densities for the model parameters. We assess the effectiveness of these spatial models based on a simulation study and a real application. We consider the prediction

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of statewide four-person family median incomes for the U.S. states based on the 1990 Current Population Survey and the 1980 Census. This application and simulation study show considerably superior performance of some of the spatial models over the regular Fay-Herriot model when good covariates remain unavailable. In some applications, some small areas are created after the completion of a survey that does not provide any direct estimates of the late-breaking unsampled small areas. Proposed spatial models generate better predictions of unsampled small area means by borrowing from neighboring residuals than the synthetic regression means that result from the regular independent random effects Fay-Herriot model. For all the spatial Bayesian models considered, their posterior distributions based on a useful class of improper prior densities on model parameters, even in the absence of data from some small areas, are shown to be proper under some mild conditions.

Keywords: Conditional autoregression; Current Population Survey; Fay-Herriot model; Intrinsic autoregression; Simultaneous autoregression; Unsampled small areas.

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1 Introduction

Sample surveys provide useful data in estimating various characteristics of a population of interest. However, when it comes to estimating a sub-population characteristic, a design-based direct estimate, based solely on data from that sub-population alone, is usually inaccurate as the accessible sample size is small and sometimes nonexistent for lack of sample. Sub-populations that lack a reasonable sample size to produce reliable direct estimates are known as small areas. Also, limited available resources often preclude many sub-populations from being selected in the sample, resulting in unsampled small areas.

To enhance accuracy of direct estimates of small areas, model-based approach has been widely used that facilitates borrowing information from such estimates of other domains and other auxiliary data. In many applications, supplementary information from other surveys and administrative data provide useful covariates. Model-based estimates are produced by suitably shrinking the direct estimates (when available) to the synthetic regression estimates based on auxiliary variables. The improvement in prediction greatly depends on to which extent the sub-population means of the characteristic are related to the auxiliary variables. When a small area has no direct estimate, the traditional independent random effects model by Fay and Herriot (1979) estimates the mean by a synthetic regression estimate alone.

Fay and Herriot (1979) proposed a very useful model to develop estimates of small area means based on direct survey estimates (if available) and synthetic regression estimates computed from auxiliary variables. Their model, which is essentially a mixed linear model, is widely known as the Fay-Herriot (FH) model in the small area estimation literature. For $i = 1, \ldots, m$, let Y_i be the direct estimate of the small area characteristic θ_i obtained from a survey. Also let $\mathbf{x}_i$ and $\boldsymbol{\beta}$ be the p-component vectors of covariates and corresponding regression coefficients, respectively. Then the FH model can be written as

$$Y_i = \theta_i + e_i, \quad \theta_i = \mathbf{x}_i^\top \beta + v_i, \quad i = 1, \ldots, m, \tag{1}$$

where sampling errors e_i, $i = 1, \ldots, m$, are independently distributed as $e_i \overset{ind}{\sim} \mathcal{N}(0, D_i)$, and are independent of random effects (small area effects) $v_i \overset{iid}{\sim} \mathcal{N}(0, \sigma_v^2)$, $i = 1, \ldots, m$. Here, the sampling variances D_i, $i = 1, \ldots, m$ are assumed to be known but the regression parameter β and model error variance σ_v^2, called model parameters, are unknown quantities. For unsampled small areas with auxiliary variables, only the second part of (1) holds for θ_i.

There has been extensive research on the FH model and its many variants. While Fay and Herriot (1979) used an empirical Bayes (EB) approach, subsequently, Prasad and Rao (1990), Datta and Lahiri (2000) and Datta et al. (2005) used the frequentist approach and derived the second-order mean squared error (MSE) of empirical best linear unbiased predictor (EBLUP) of θ_i and various second-order approximate unbiased estimators of the MSE's (see Datta and Lahiri, 2000). Earlier Ghosh (1992) proposed a hierarchical Bayesian (HB) approach to the Fay-Herriot model. In the Bayesian framework, the FH model in (1) can be expressed as the following HB model:

$$Y_i|\theta_1, \ldots, \theta_m, \boldsymbol{\beta}, \sigma_v^2 \overset{ind}{\sim} \mathcal{N}(\theta_i, D_i), \quad i = 1, \ldots, m, \tag{2}$$

$$\theta_i|\boldsymbol{\beta}, \sigma_v^2 \overset{ind}{\sim} \mathcal{N}(\mathbf{x}_i^\top \boldsymbol{\beta}, \sigma_v^2), \quad i = 1, \ldots, m, \tag{3}$$

$$\pi(\boldsymbol{\beta}, \sigma_v^2) \propto g(\boldsymbol{\beta}, \sigma_v^2), \tag{4}$$

where $g(\cdot)$ is a suitably chosen function of β and σ_v^2, which expresses a prior probability density function (pdf) for these parameter. Without specifying a prior pdf as in (4), an EB predictor for θ_i was originally developed by Fay and Herriot (1979). While a standard EB approach usually underestimates the measure of uncertainty of the EB estimator of θ_i, the HB approach facilitates uncertainty quantification due to estimation of unknown model parameters, $\boldsymbol{\beta}$ and σ_v^2. The uncertainty is fully captured by the posterior distribution of the model parameters.

In model-based estimation, random effects are of great importance in capturing the remaining variability of the θ_i's that is not explained by the regression model. In real applications, small areas generally involve features such as population size, ethnicity, age-group and education level, which might affect the variability of small area effects. Furthermore, when disease prevalence rates are of interest, it is reasonable to assume that random effects of adjacent small areas are correlated in a certain way. In such cases, the FH model given in (1), which we refer to as the independent FH random-effects model, oversimplifies and misspecifies the distribution of random effects by assuming a common and independent distribution. Although benefits from model-based approach are substantial, it is known that it can perform poorly under model misspecification as the domain sample size increases (Rao and Molina, 2015).

In Section 2, we propose small area estimation models which effectively account for heteroscedasticity and spatial dependence of the small area effects. In particular, we take a fully Bayesian approach by specifying a class of noninformative priors on the model parameters. Spatial dependence of small area effects are modeled by four widely used autocorrelation structures. These include conditional autoregressive (CAR), simultaneous autoregressive (SAR), intrinsic autoregressive (IAR) and a spatial model suggested by Rao and Molina (2015) which we refer to as the SRM model. There is an abundance of literature on spatial models under the Bayesian framework. Sun et al. (1999) studied an HB model with the conditional and intrinsic autoregressive models on the random effects. The same models were considered by Speckman and Sun (2003) in the context of Bayesian spline smoothing. For small area estimation, You and Zhou (2011) modeled small area effects using a conditional autoregressive model. As an extension of the time series FH model (Datta et al., 1999), Torabi (2012) proposed a spatio-temporal model with intrinsic autoregressive random effects. Porter et al. (2014) proposed an extension of the FH model with functional covariates and intrinsic autoregressive random effects. Porter et al. (2015) incorporated the conditional autoregressive random effects on the multivariate FH model.

We note that the existing Bayesian spatial FH models consider a proper prior on

σ_v^2. Furthermore, most of the models assume a conditional autoregressive structure on the random effects. The main contributions of this paper are as follows. First, we provide in Section 2 sufficient conditions for posterior propriety for a class of improper noninformative priors on model parameters. These conditions do not depend on the assumed autocorrelation structure provided that it yields a positive definite covariance matrix for the random effects. Secondly, in Subsection 2.1 we further extend the spatial models to estimate small area means of several unsampled small areas with no direct estimates. The unsampled area mean θ_i is estimated by borrowing strength from the auxiliary variables of this area and, for spatial models, from the regression residuals of its neighboring areas. We again prove that the posterior distributions based on improper noninformative priors are proper for this more practical problem. The effectiveness of the proposed spatial models is demonstrated in Section 3 and 4. We apply the spatial models to simulated data and real data sets from the Current Population Survey (CPS). We compare various spatial models in Section 4 to estimate four-person family median incomes for the forty-nine contiguous states of U.S. based on the CPS data and appropriate covariates from previous census and administrative data. Our data analysis and simulation studies reveal that proposed spatial models significantly improve prediction accuracy and reduce measure of uncertainty (posterior variance). We provide concluding remarks in Section 5. All technical discussions including the proofs of propriety of the various posterior distributions are provided in Section 6.

2 Some spatial alternatives to the independent FH model

Let $\mathbf{Y} = (Y_1, \ldots, Y_m)^\top$ be the m-component vector with the direct estimates of m small areas, and $\mathbf{D} = \text{diag}\{D_i\}_{i=1}^m$ be the $m \times m$ diagonal matrix with the sampling variances of the direct estimates. We denote by $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_m)^\top$ the m-component vector of small area means. Also, let $\mathbf{X} = [\mathbf{x}_1, \cdots, \mathbf{x}_m]^\top$ be the $m \times p$ matrix of auxiliary variables, where $\mathbf{x}_i$ is the p-component vector of auxiliary variables (including the intercept term) for the ith small area. A special case of the HB model given in (2)–(4) can be expressed as

$$\mathbf{Y}|\boldsymbol{\theta}, \boldsymbol{\beta}, \sigma_v^2 \sim \mathcal{N}_m(\boldsymbol{\theta}, \mathbf{D}), \tag{5}$$

$$\boldsymbol{\theta}|\boldsymbol{\beta}, \sigma_v^2 \sim \mathcal{N}_m(\mathbf{X}\boldsymbol{\beta}, \sigma_v^2\mathbf{I}_m), \tag{6}$$

$$\pi(\boldsymbol{\beta}, \sigma_v^2) \propto 1, \tag{7}$$

where $\boldsymbol{\beta}$ is the p-component regression coefficient vector, σ_v^2 is the model error variance and $\mathbf{I}_m$ is the identity matrix of order m. The uniform prior (7) on the model parameters is a popularly used noninformative prior. The resulting posterior pdf is proper provided that $m > p + 2$. See Berger (1985) and Datta and Smith (2003).

From (6), it can be seen that θ_i, $i = 1, \ldots, m$, are independently distributed with common random effects variance σ_v^2 over the small areas. However, in many cases, the area characteristic of interest is closely related to geographical factors such as population size, ethnicity, age-group and education level. In such cases, due to misspecified random effects distribution (independence and equal variance, in this case), inference based on the hierarchical model (5)–(7) may produce unreliable estimates which can yield erroneous decisions. Moreover, if available auxiliary variables are not sufficiently related to the small area means or lurking variables exist, this problem can be even worse as they cause extra variability that cannot be explained by the independently and identically distributed (i.i.d.) random effects.

Let $\mathbf{W} = \{w_{ij}\}_{ij}$, $1 \leq i, j \leq m$, be the adjacency matrix which plays an important role in capturing spatial dependency. In particular, $w_{ij} = 1$ if ith and jth small areas are connected via geographical boundary or through other mechanisms (for example, air traffic), and $w_{ij} = 0$, otherwise. Also, $w_{ii} = 0$ for $i = 1, \ldots, m$. Note that w_{ij}'s need not be binary; they can take other positive values, such as the "length" of the geographical border or volumes of air traffic between the two areas. Since $\mathbf{W}$ is a symmetric, non-null matrix, its eigenvalues λ_i's are real with at least one is non-zero. We denote the ith largest eigenvalue by λ_i such that $\lambda_1 \geq \ldots \geq \lambda_m$. Since $\sum_{i=1}^m w_{ii} = 0$ we get as a result that $\lambda_m < 0 < \lambda_1$. Let $w_{i.} = \sum_{j=1}^m w_{ij}$ be the sum of the ith row of W and define $\mathbf{L} = \text{diag}\{w_{i.}\}_{i=1}^m$. We assume that diagonal elements of $\mathbf{L}$ are positive and we define $\widetilde{\mathbf{W}} = \mathbf{L}^{-1}\mathbf{W}$. We consider four different spatial dependencies that are represented by the following positive definite "precision" matrices associated with random effects:

$$\text{CAR: } \boldsymbol{\Omega}_2(\rho) = \mathbf{I}_m - \rho\mathbf{W}, \qquad \rho \in (\lambda_m^{-1}, \lambda_1^{-1}), \tag{8}$$

$$\text{SAR: } \boldsymbol{\Omega}_3(\rho) = (\mathbf{I}_m - \rho\widetilde{\mathbf{W}})^\top(\mathbf{I}_m - \rho\widetilde{\mathbf{W}}), \qquad \rho \in (-1, 1), \tag{9}$$

$$\text{IAR: } \boldsymbol{\Omega}_4(\rho) = \mathbf{L} - \rho\mathbf{W}, \qquad \rho \in (-1, 1), \tag{10}$$

$$\text{SRM: } \boldsymbol{\Omega}_5(\rho) = \rho\mathbf{R} + (1 - \rho)\mathbf{I}_m, \qquad \rho \in (0, 1), \tag{11}$$

where the model parameter ρ is called the spatial autocorrelation that captures the strength of spatial dependence. The matrix $\mathbf{R}$ above is defined as $\mathbf{R} = \mathbf{L} - \mathbf{W}$. The ith diagonal element of $\mathbf{R}$ is the number of neighborhoods of the ith small area, and the (i, j)th off-diagonal element is -1 if the ith and the jth small areas are connected and 0

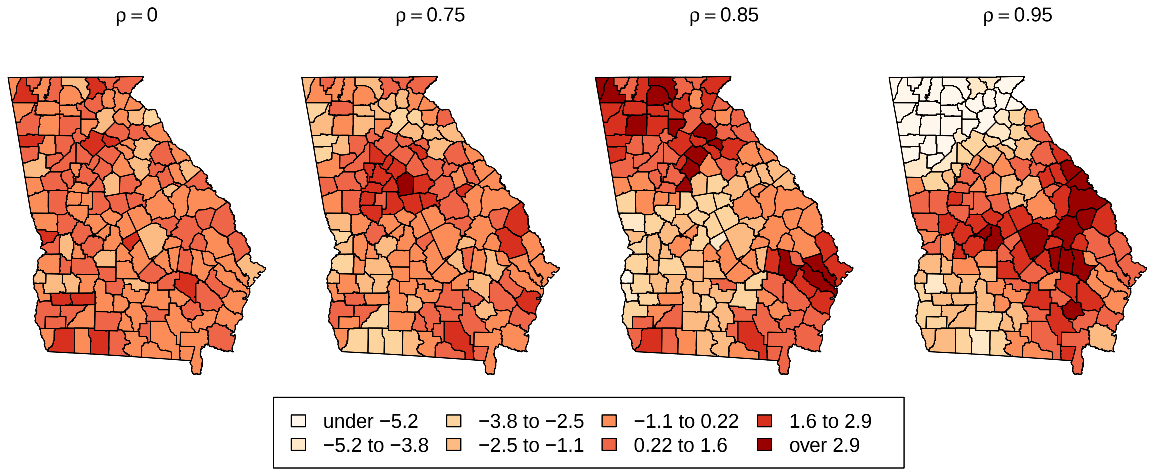

Figure 2.1: Geographical illustrations of spatial dependencies with various values of ρ. Each geographical region is one of 159 counties of Georgia. Data are generated from the SAR model.

otherwise. The SAR adjacency matrix $\widetilde{\mathbf{W}}$ is row-normalized so that as ρ varies between -1 to 1, the precision matrix remains positive definite. The SRM precision matrix is a convex combination of the extreme IAR precision matrix $\mathbf{\Omega}_4(1)$ and independent FH precision matrix (the coefficient ρ used in the convex combination is different from the same symbol used in $\mathbf{\Omega}_4(\rho)$). Note that even if the diagonal elements of a precision matrix are all equal, for example the CAR model, the diagonal elements of the inverse may not be all equal, leading to heteroscedasticity of random effects.

Figure 2.1 graphically illustrates the strength of spatial autocorrelation, where the small areas are the $m = 159$ counties of Georgia. Data are generated from $\mathcal{N}_m(\mathbf{0}_m, \sigma_v^2\{\mathbf{\Omega}_3(\rho)\}^{-1})$, where $\mathbf{0}_m$ is the m-component null vector, $\sigma_v^2 = 1$ and $\rho = 0, 0.75, 0.85, 0.95$. Although a value of ρ is not directly comparable from model to model, a large value of it represents strong neighborhood similarity.

A CAR model assumes that θ_i depends only on neighboring small area means. In other words, θ_i is correlated with other areas only through area means of surrounding areas. Similar interpretation holds for IAR and SRM models. On the other hand, a SAR model assumes that θ_i depends simultaneously on other θ_j, $j \neq i$, but have larger (weaker) correlations for adjacent (distant) areas. The independent FH model can be viewed as a special case of the above class of models with $\rho = 0$. For convenience of notation, the precision matrix for the independent FH model is denoted by $\mathbf{\Omega}_1(\rho) = \mathbf{I}_m$, which actually does not depend on ρ.

We consider the following five HB models with $k = 1, \ldots, 5$ ($\boldsymbol{\Omega}_1(\rho) = \mathbf{I}_m$):

$$\mathbf{Y}|\boldsymbol{\theta}, \boldsymbol{\beta}, \sigma_v^2, \rho \sim \mathcal{N}_m(\boldsymbol{\theta}, \mathbf{D}), \tag{12}$$

$$\boldsymbol{\theta}|\boldsymbol{\beta}, \sigma_v^2, \rho \sim \mathcal{N}_m(\mathbf{X}\boldsymbol{\beta}, \sigma_v^2\{\boldsymbol{\Omega}_k(\rho)\}^{-1}), \tag{13}$$

$$\pi(\boldsymbol{\beta}, \sigma_v^2, \rho) \propto g(\sigma_v^2)h(\rho), \quad \boldsymbol{\beta} \in \mathbb{R}^p,\ \sigma_v^2 > 0,\ \ell_k < \rho < u_k, \tag{14}$$

where $g(\sigma_v^2)$ and $h(\rho)$ are suitable functions of σ_v^2 and ρ, and ℓ_k and u_k are the lower and upper bounds of ρ under the kth model. Let $1(\cdot)$ be the indicator function taking the value 1 when its argument is true and 0 otherwise. Then the (joint) posterior pdf of model parameters is proper under the following conditions.

Theorem 1. For all the HB models given in (12)–(14), the posterior probability density functions are proper if the following conditions hold for some positive constant $c > 0$:

(a) $\int_0^\infty g(\sigma_v^2)1(\sigma_v^2 \le c)d\sigma_v^2 < \infty$.

(b) $\int_0^\infty (\sigma_v^2)^{-(m-p)/2} g(\sigma_v^2)1(\sigma_v^2 > c)d\sigma_v^2 < \infty$.

(c) $\int_{\ell_k}^{u_k} h(\rho)d\rho < \infty$.

Any bounded function of ρ satisfies (c) in Theorem 1 as the domains are all bounded. Consider the following family of noninformative priors:

$$\pi(\boldsymbol{\beta}, \sigma_v^2, \rho) \propto (\sigma_v^2)^{-\alpha}1(\ell_k < \rho < u_k), \quad \boldsymbol{\beta} \in \mathbb{R}^p,\ \sigma_v^2 > 0. \tag{15}$$

Under (15), we provide the conditions for the posterior propriety in the following corollary.

Corollary 1.1. For any of the hierarchical Bayes models given in (12)–(13) with the prior in (15), the posterior pdf is proper if and only if $1 - (m-p)/2 < \alpha < 1$.

2.1 Estimation of unsampled small area means

In this section, we consider the case when there are several unsampled small areas that have no direct estimates. Since the direct estimate of an unsampled area is missing, the prediction of the mean of the unsampled small area is solely based on its synthetic estimator, and the vector of regression residuals, with more emphasis on the components of its neighboring areas. Here, we propose to exploit spatial dependencies in predicting area means of unsampled small areas. Without loss of generality, let there be m_1 unsampled small areas and $Y_{m_1+1}, \ldots, Y_m$ be the direct estimates of the sampled small areas. Based on the $m_2 = m - m_1$ sampled direct estimates, we consider the following HB models:

$$\mathbf{Y}_{(2)}|\boldsymbol{\theta}, \boldsymbol{\beta}, \sigma_v^2, \rho \sim \mathcal{N}_{m_2}(\boldsymbol{\theta}_{(2)}, \mathbf{D}_{(2)}), \tag{16}$$

$$\boldsymbol{\theta}|\boldsymbol{\beta}, \sigma_v^2, \rho \sim N_m(\mathbf{X}\boldsymbol{\beta}, \sigma_v^2\{\boldsymbol{\Omega}_k(\rho)\}^{-1}), \quad k = 1, \ldots, 5, \tag{17}$$

$$\pi(\boldsymbol{\beta}, \sigma_v^2, \rho) \propto (\sigma_v^2)^{-\alpha} 1(\ell_k < \rho < u_k) \quad \text{for } \boldsymbol{\beta} \in \mathbb{R}^p, 0 < \sigma_v^2, \tag{18}$$

where $\mathbf{Y}_{(2)} = (Y_{m_1+1}, \ldots, Y_m)^\top$, $\mathbf{D}_{(2)} = \text{diag}\{D_i\}_{i=m_1+1}^m$ and $\boldsymbol{\theta}_{(2)} = (\theta_{m_1+1}, \ldots, \theta_m)^\top$ is the subvector of θ corrosponding to the sampled areas. Then the posterior pdf for this model is proper under the following conditions.

Theorem 2. Under the various hierarchical Bayes models given in (16)–(18), the corresponding posterior pdf is proper if $1 - (m - m_1 - p)/2 < \alpha < 1$.

We can directly see that equivalent conditions for the posterior propriety are $\alpha < 1$ and $m - p - 2 + 2\alpha > m_1$. Thus, using the uniform prior, $\alpha = 0$, the posterior distribution is proper as long as the number of unsampled small areas is fewer than $m - p - 2$.

Remark 1. In all applications, we use the prior corresponding to $\alpha = 0$, which is a uniform prior for the model parameters. In a recent article, Berger et al. (2020) suggested a class of objective priors for model parameters in general normal hierarchical models. According to them, uniform priors for the model parameters do not produce optimal solution in terms of admissibility. Following them, $\pi(\sigma_v^2) = (\sigma_v^2)^{(-.5)}$ is a better alternative for σ_v^2, and $\pi(\boldsymbol{\beta}) = [m + \boldsymbol{\beta}^T\mathbf{X}^T\mathbf{D}^{-1}\mathbf{X}\boldsymbol{\beta}]^{-(p-1)/2}$ may be a better alternative for $\boldsymbol{\beta}$. For $p = 1$, the prior of $\boldsymbol{\beta}$ is uniform. Since $\pi(\boldsymbol{\beta})$ is bounded above by 1, it can be easily proved that this general improper prior for $\boldsymbol{\beta}$ also yields a proper posterior density.

3 A simulation study

In this section, we evaluate the prediction performances of independent FH model and the four spatial models in the absence of "good" covariates using simulated data sets. Small areas of interest are $m = 159$ counties in the state of Georgia. Sampling variances D_i, $i = 1, \ldots, m$, are independently generated from a gamma distribution with mean 3.3 and shape parameter 1.1 (which lead to a scale 3). We consider two independent covariates x_1 and x_2, each exhibiting a spatial dependence modeled by

$$\mathbf{x}_j \overset{iid}{\sim} \mathcal{N}_m(\mathbf{0}_m, \{\boldsymbol{\Omega}_3(\rho)\}^{-1}), \quad j = 1, 2.$$

We consider 16 different simulation settings corresponding to various values of spatial autocorrelation and model error variance. Specifically, we consider four different degrees of spatial autocorrelation, $\rho = 0, 0.75, 0.85, 0.95$. For the model error variance parameter σ_v^2, we also consider four values, $\sigma_v^2 = \bar{D}/8, \bar{D}/2, \bar{D}, 2\bar{D}$, where $\bar{D} = m^{-1}\sum_{i=1}^m D_i$.

For given values of ρ, σ_v^2 and D_i, $i = 1, \ldots, m$, we consider $S = 50$ replicates of data sets generated from

$$\theta_i \overset{ind}{\sim} \mathcal{N}(\mu_i, \sigma_v^2), \quad Y_i|\theta_i \overset{ind}{\sim} \mathcal{N}(\theta_i, D_i), \quad i = 1, \ldots, m,$$

where $\mu_i = \beta_1 x_{i1} + \beta_2 x_{i2}$ and $(\beta_1, \beta_2)^\top = (2, 0.5)^\top$. This makes $\mathbf{x}_1$ a more influential covariate than $\mathbf{x}_2$. Also, both variables will introduce spatial variation to the θ_i's.

To examine how the spatial models can capture extra variability introduced by the spatial dependence from one (or both) of the missing covariates, we consider $L = 3$ different reasonable combinations of covariates. In particular, we consider $\mathbf{X}_1 = \mathbf{1}_m$, $\mathbf{X}_2 = [\mathbf{1}_m, \mathbf{x}_2]$ and $\mathbf{X}_3 = [\mathbf{1}_m, \mathbf{x}_1]$, where $\mathbf{1}_m$ represents the m-component vector of ones. We do not consider the full model involving both the covariates since that model will fully capture μ_i, consequently, the independent FH model will be sufficient to capture the variability of the i.i.d. random effects. For each $k = 1, \ldots, 5$, we fit the following HB models:

$$\mathbf{Y}|\boldsymbol{\theta}, \boldsymbol{\beta}_\ell, \sigma_v^2, \rho \sim \mathcal{N}_m(\boldsymbol{\theta}, \mathbf{D}), \tag{19}$$

$$\boldsymbol{\theta}|\boldsymbol{\beta}_\ell, \sigma_v^2, \rho \sim \mathcal{N}_m(\mathbf{X}_\ell\boldsymbol{\beta}_\ell, \sigma_v^2\{\boldsymbol{\Omega}_k(\rho)\}^{-1}), \quad \ell = 1, \ldots, 3, \tag{20}$$

$$\pi(\boldsymbol{\beta}_\ell, \sigma_v^2, \rho) \propto 1(\ell_k < \rho < u_k), \tag{21}$$

where the dimensions of $\boldsymbol{\beta}_\ell$ agree with the corresponding $\mathbf{X}_\ell$.

Let $\hat{\boldsymbol{\theta}}_k = (\hat{\theta}_{k1} \ldots, \hat{\theta}_{km})^\top$ be the predictor under the kth model. For convenience of notation we denote the vector of direct estimates $\mathbf{Y}$ by $\hat{\boldsymbol{\theta}}_0$. To evaluate predictors, $\hat{\boldsymbol{\theta}}_k$, $k = 0, \ldots, 5$, in terms of prediction accuracy, we calculate their total squared prediction error, $\text{TSPE}_k = \sum_{i=1}^m (\hat{\theta}_{ki} - \theta_i)^2$, after fitting the models above for each replicated data set. We then average TSPE_k over the replicated data sets for each $k = 0, \ldots, 5$, to compute total empirical mean squared prediction error, TeMSPE_k, given by $\text{TeMSPE}_k = S^{-1}\sum_{s=1}^S \sum_{i=1}^m (\hat{\theta}_{ki}^{(s)} - \theta_i^{(s)})^2$. Here, $\theta_i^{(s)}$ is the true value of the ith small area mean based on the sth replicated data set; and $\hat{\theta}_{ki}^{(s)}$ is the predicted value of $\theta_i^{(s)}$ by the kth model. By setting the direct estimate's TeMSPE_0 as the baseline, the independent FH and the

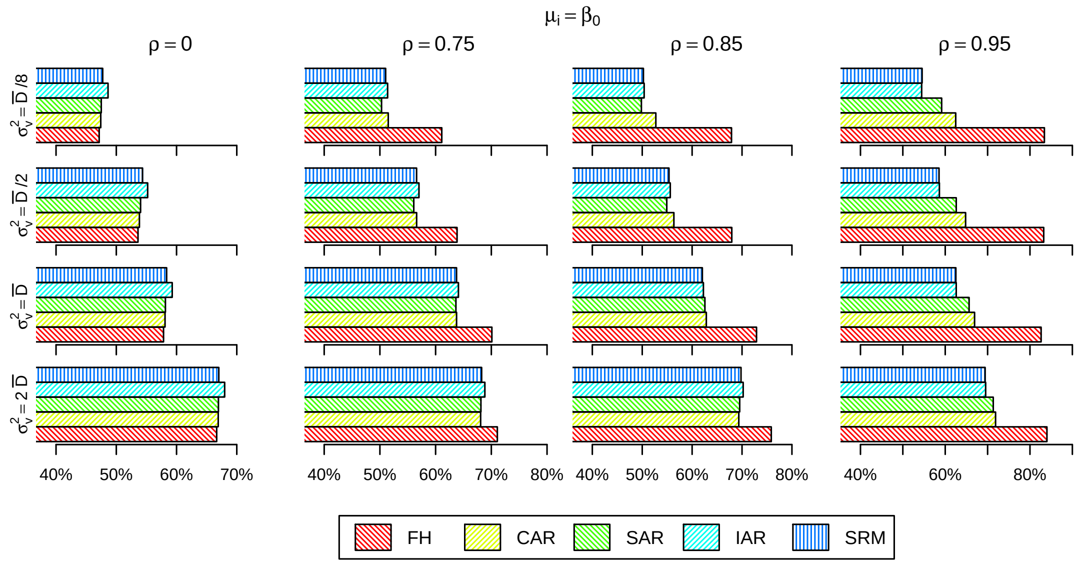

Figure 3.1: Prediction accuracy improvements when no covariate is in the fitted model. The height of each horizontal bar represents the kth model's total empirical mean squared prediction error, TeMSPE_k, as a fraction of direct estimate's total empirical mean squared prediction error, TeMSPE_0. Data are generated under four different levels of model error variance and spatial correlation.

spatial models are assessed by the ratio

$$\text{Frc}_{k,0} = \frac{\text{TeMSPE}_k}{\text{TeMSPE}_0},$$

which expresses TeMSPE_k as a fraction of TeMSPE_0, the baseline total empirical mean squared prediction error. A better model will have a smaller value of $\text{Frc}_{k,0}$. Simulation results for the three reasonable candidate models based on combinations of covariates in the fitted model are summarized in Figures 3.1 – 3.3, respectively. In each figure, a horizontal bar is drawn for $\text{Frc}_{k,0}$, where a shorter bar for a model means smaller total mean squared prediction error for that model relative to the vector of direct estimates Y.

From Figures 3.1 to 3.3, when $\rho = 0$ the μ_i's and hence the θ_i's do not have any spatial variation whether the model error terms v_i's are small (for small σ_v^2) or large (for large σ_v^2), we find that all the spatial models are as useful as the independent FH model in terms of total mean squared prediction error. These show no loss in using a spatial model (other than any extra effort to fit a slightly complicated model) for predicting small area means that have no spatial variation.

Figure 3.1 illustrates results with no covariate (intercept only) in the fitted models,

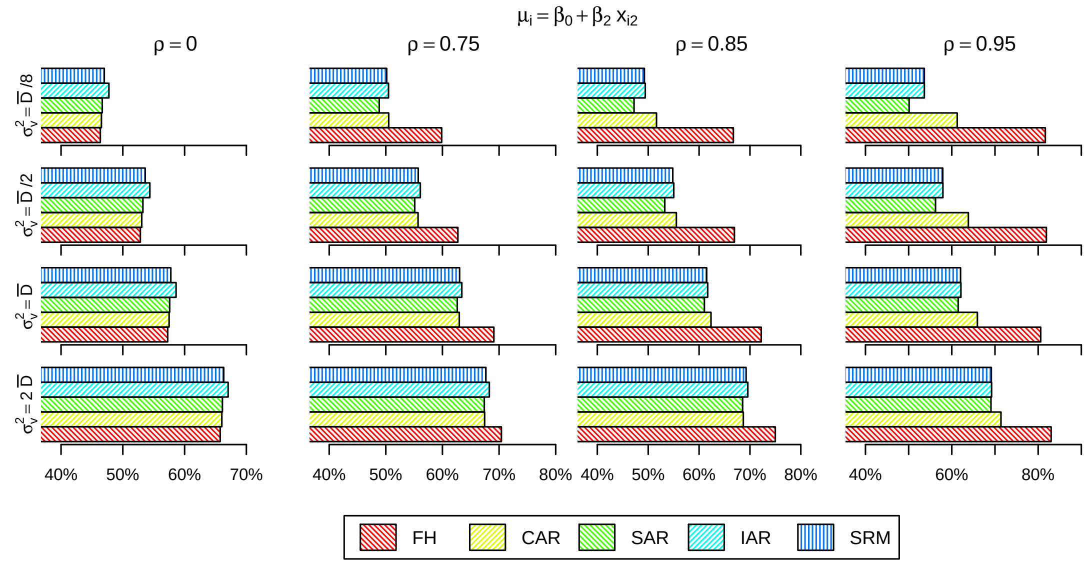

Figure 3.2: Prediction accuracy improvements when only the covariate x_2 is in the fitted model. The height of each horizontal bar represents the kth model's total empirical mean squared prediction error, TeMSPE_k, as a fraction of direct estimate's total empirical mean squared prediction error, TeMSPE_0. Data are generated under four different levels of model error variance and spatial correlation.

for four values of spatial correlation and four values of model error variance. As the spatial correlation increases, we see from the leftmost column to the rightmost one that the performances of all the spatial models are better than the independent FH model, and they improve progressively. In the case of very strong spatial pattern with $\rho = 0.95$, the FH model deteriorates significantly with its predictions that are very similar to the direct estimates. While the spatial models have much smaller TeMSP relative to the FH model for smaller model error variance (top rows of the figure), their superior performance persists, though gets diminished for larger model error variance (in case of a very large model error variance all the models produce little shrinkage of the direct estimates). The average improvements of the spatial models over the FH independent model range from 33% ($\sigma_v^2 = \bar{D}/8$) to 15% ($\sigma_v^2 = 2\bar{D}$).

Figure 3.2 presents results when only the less influential covariate $\mathbf{x}_2$ is used in fitting all five models, again using the same four values of spatial correlation and four values of model error variance. The results for this setup are practically identical to those in Figure 3.1 with an intercept only model. Thus the inclusion of a weaker covariate displaying the same spatial pattern as in the μ_i's did not improve the performance of the independent FH model. The spatial models are found to be better suited to capture

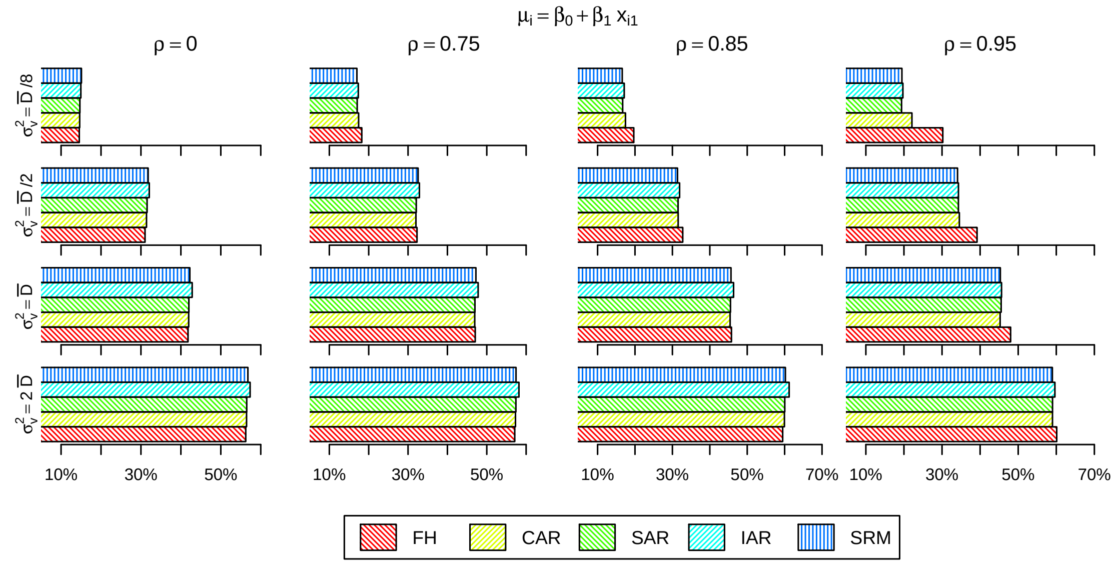

Figure 3.3: Prediction accuracy improvements when only the covariate x_1 is in the fitted model. The height of each horizontal bar represents the kth model's total empirical mean squared prediction error, TeMSPE_k, as a fraction of direct estimate's total empirical mean squared prediction error, TeMSPE_0. Data are generated under four different levels of model error variance and spatial correlation.

the spatial pattern of the μ_i's in the absence of the most effective covariate. Among the spatial models, the SAR model shows the largest improvement which can be attributed to the SAR pattern used in generating the μ_i's.

Figure 3.3 presents results when the most influential covariate $\mathbf{x}_1$ in addition to the intercept is used in fitting all five models for the identical combinations of spatial correlation and model error variance. The results for this setup show that the inclusion of this covariate that mostly determines the spatial pattern of μ_i's makes up most of the deficiency of the independent FH model relative to the spatial models (still incurs about 50% more TeMSP over the spatial models when $\sigma_v^2 = \bar{D}/8$ and $\rho = 0.95$). Also, the inclusion of the more influential predictor leads to considerably improved predictive performance for all five models, measured by their TeMSP values.

Figure 3.4 illustrates spatial models' average posterior standard deviations as fractions of the average sampling standard deviation, $m^{-1}\sum_{i=1}^{m}\sqrt{D_i}$, when only the less influential covariate $\mathbf{x}_2$ is included in the fitted models. The results show a similar pattern as in Figure 3.3. All five models improve the variability of predictions over the direct estimates. Compared with the independent FH model, predictions based on spatial models show smaller variability when $\rho > 0$. As the spatial pattern strengthens, the

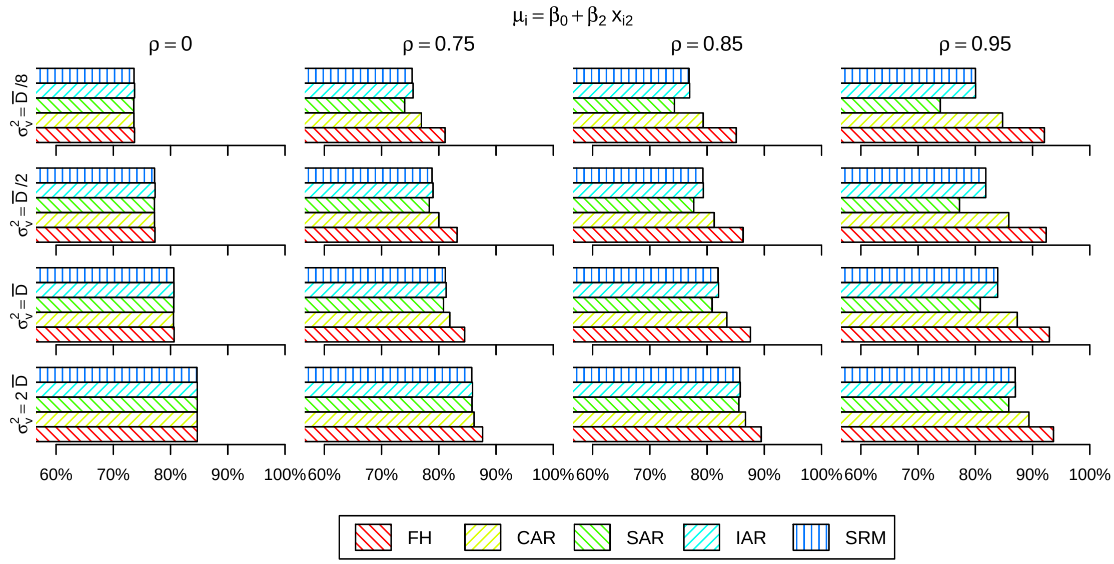

Figure 3.4: Ratio of average posterior standard deviation to average sampling standard deviation when only the covariate x_2 is in the fitted model. Data are generated under four different levels of model error variance and spatial correlation.

average posterior standard deviation of small area means based on the SAR model stays approximately the same as this model more effectively captures spatial dependence of small area means. On the other hand, the average posterior standard deviations of other models increase as ρ increases, whereas spatial models still outperform the independent FH model. When $\rho = 0.95$, the improvements made by the independent FH model are less than 10%. When there is no spatial pattern in the residual, $\rho = 0$, the variabilities of all five models are practically identical. The average posterior standard deviations corresponding to design matrices $\mathbf{X}_1$ and $\mathbf{X}_3$ are summarized in Figures A.1 and A.2 in Appendix.

We find from our simulations that if any spatial pattern exists in the residuals after fitting a model, then the spatial models effectively capture the extra variability and allow more accurate predictions. Although we generate spatial patterns in the μ_i's from the SAR model, the SRM and the IAR models occasionally display competitive performance in terms of TeMSPE. When no spatial pattern exists, spatial models show the same performace as the independent FH model in terms of total emprical mean squared prediction error and average posterior standard deviation.

4 An application to some current population survey data

In this section, we evaluate the spatial models in terms of their accuracy of prediction of state level small area means of forty-nine U.S. states. The U.S. Department of Health and Human Service (HHS) annually needed accurate data for median income of four-person families in a state to implement a welfare program. While accurate national median income data are available from the Current Population Survey (CPS), the CPS data do not provide accurate median income data for the individual states. To supply accurate statistics to the HHS, the U.S. Census Bureau considered model-based small area estimation method by utilizing auxiliary data from other federal programs. We apply our proposed spatial models to estimate four-person family median incomes for the contiguous forty-nine U.S. states (including the Washington, D.C.) for the year 1989. We consider 1989 so that we can compare our predictions with the more reliable corresponding statistics obtained from data compiled from the 1990 Census long form. Prediction performances are measured using data from all small areas, and suitable subsets of data after leaving out data for some states to treat these states as unsampled small areas.

4.1 Four-person family median income estimation

Let θ_i be the true four-person family median income of the ith state for the year 1989, where $i = 1, \ldots, m$ and $m = 49$ is the number of contiguous states including the District of Columbia. The states of Alaska and Hawaii are excluded as they are not geographically connected to the mainland. Let Y_i be the direct estimate of θ_i based on the 1990 CPS that collected income data for 1989. Covariates of interest are 1980 census median income x_{i1} and an adjusted 1980 census median income x_{i2}. The adjusted census median income x_{i2} is defined as $(\mathrm{PCI}_{i,1989}/\mathrm{PCI}_{i,1979})x_{i1}$, $i = 1, \ldots, m$, where $\mathrm{PCI}_{i,1979}$ and $\mathrm{PCI}_{i,1989}$ are the 1979 and 1989 per capita income of the ith state provided by the Bureau of Economic Analysis of the U.S. Department of Commerce. It has been known that the adjusted census median income is a good covariate which accounts for the variability of the small area median income. We consider the state level median incomes obtained from the 1990 census as the true values of θ_i, $i = 1, \ldots, m$, for the year 1989. We note that dollar amounts are scaled by $1000.

We fit all five models as described by (12)–(14) with $\mathbf{X} = [\mathbf{1}_m, \mathbf{x}_1, \mathbf{x}_2]$ and $\mathbf{X} = [\mathbf{1}_m, \mathbf{x}_1]$. For the latter, we exclude the adjusted census median income from the fitted

	$\mu_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2}$				$\mu_i = \beta_0 + \beta_1 x_{i1}$		
	β_0	β_1	β_2	ρ	β_0	β_1	ρ
FH	1.41 (5.58)	0.39 (0.37)	0.67 (0.12)	NA	-3.86 (7.29)	1.97 (0.33)	NA
CAR	1.31 (5.53)	0.40 (0.37)	0.67 (0.12)	-0.06/0.038 (0.14)	-1.11 (7.44)	1.88 (0.33)	0.14/0.17 (0.04)
SAR	1.51 (5.83)	0.44 (0.39)	0.64 (0.13)	0.10/0.396 (0.48)	-2.59 (9.14)	1.92 (0.29)	0.76/0.80 (0.14)
IAR	1.35 (5.58)	0.45 (0.40)	0.64 (0.13)	0.21/0.831 (0.55)	-2.85 (6.99)	1.91 (0.29)	0.93/0.99 (0.09)
SRM	2.19 (6.00)	0.50 (0.44)	0.59 (0.15)	0.57/0.803 (0.27)	-2.04 (7.83)	1.89 (0.29)	0.85/0.97 (0.13)

Table 1: Posterior mean and standard deviation (in parentheses) of model parameters under the independent FH model and four spatial models. For spatial autocorrelations, the posterior mode is also provided beside the posterior mean. The left part of the table summarizes posterior distributions of model parameters when both covariates are used in the fitted model, and the right side of the table summarizes when only x_1 is included in the fitted model.

model. The prior distribution for the model parameters is the noninformative prior (15) with $\alpha = 0$. Table 1 summarizes the posterior distributions of model parameters in terms of the posterior mean, mode, and standard deviation. All models provide nearly similar posterior distributions of regression coefficients. When both the covariates are included, all models show that β_2 is the only significant regression coefficient and β_1 is insignificant as its 95% credible interval includes zero (not presented here). The posterior distributions of ρ indicate no strong spatial dependency, whereas posterior modes indicate a mild degree of spatial pattern (the SRM model is an exception). On the other hand, when only x_1 is used in the regression model, β_1 becomes significant, and the posterior distribution of ρ is concentrated away from zero indicating a strong spatial pattern for all the spatial models, including the CAR model. We emphasize that for the CAR model, the permissible range of the spatial parameter ρ is $(-0.349, 0.185)$.

We use the estimates from the fitted models and the true values from the 1990 census data, and calculate the squared deviations and obtain the ASD by averaging the 49 squared deviations for each model. We also obtain posterior standard deviations associated with $\hat{\theta}_{ki}$ and calculate average posterior standard deviations (APSD) for each $k = 1, \ldots, 5$. Table 2 lists ASD, APSD and respective percentage improvements. When both the covariates are available, the SRM model has approximately 13% smaller ASD and 4% smaller APSD than the independent FH model. In terms of ASD, the second best performing model is the SAR having approximately 11% smaller ASD. In terms of APSD, the IAR is the second best model with 2% smaller APSD. When only x_1 is included in model fitting (the right half of the Table), the SAR model has approximately 40% smaller ASD and 15% smaller average posterior standard deviation than the independent FH model. The IAR and SRM models show reasonably good performances having approximately 35% smaller ASD and 15% smaller APSD over the independent FH model.

	$\mu_i = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2}$				$\mu_i = \beta_0 + \beta_1 x_{i1}$			
	ASD	ASD-PI	APSD	APSD-PI	ASD	ASD-PI	APSD	APSD-PI
FH	2.899	0%	1.940	0%	7.241	0%	2.317	0%
CAR	2.989	-3.12%	1.948	-0.40%	5.643	22.07%	2.232	3.66%
SAR	2.583	10.92%	1.937	0.15%	4.318	40.37%	1.963	15.27%
IAR	2.642	8.89%	1.903	1.94%	4.614	36.28%	1.994	13.95%
SRM	2.507	13.51%	1.855	4.41%	4.554	37.12%	1.978	14.62%

Table 2: Average squared deviations and average posterior standard deviations when all covariates are available, and only less informative covariate, x_1, is available, respectively. The percentage improvements (PI) over the independent FH model are listed.

By removing the covariate $\mathbf{x}_2$ from the full model, the ASD for the SAR model increases approximately 66%, and the ASD for the independent FH model increases more than 140%. This example shows the effectiveness of spatial models over the independent FH model when good covariates are unavailable.

4.2 Estimation of some unsampled state means excluding their CPS values

In this section, we evaluate spatial models in terms of unsampled small area prediction accuracy. At each instance, we arbitrarily exclude direct estimates of multiple states and make predictions for the median incomes of the excluded states. For our 49 states, we have 12 data sets with $m_1 = 4$ or $m_1 = 5$ missing states, where m_1 is the number of small areas given in Section 2.1, whose CPS estimates are excluded from fitting the model. Excluded states for these twelve data sets are listed in Table 3. For convenience of notation, we denote the median income and corresponding direct estimate of the jth unsampled area by θ_j and Y_j, $j = 1, \ldots, m_1$.

Set 1				Set 2				Set 3				
MS	OK	AR	SD	DE	AZ	CO	TN	WV	MD	NV	MI	
Set 4				Set 5				Set 6				
NC	MT	NY	NE	ID	ND	DC	GA	MO	VT	WY	AL	
Set 7				Set 8				Set 9				
LA	UT	FL	WA	MN	MA	TX	SC	KY	RI	VA	WI	
Set 10				Set 11				Set 12				
IN	IL	PA	NH	CA	ME	OH	NJ	OR	KS	NM	IA	CT

Table 3: Excluded small areas for each data set.

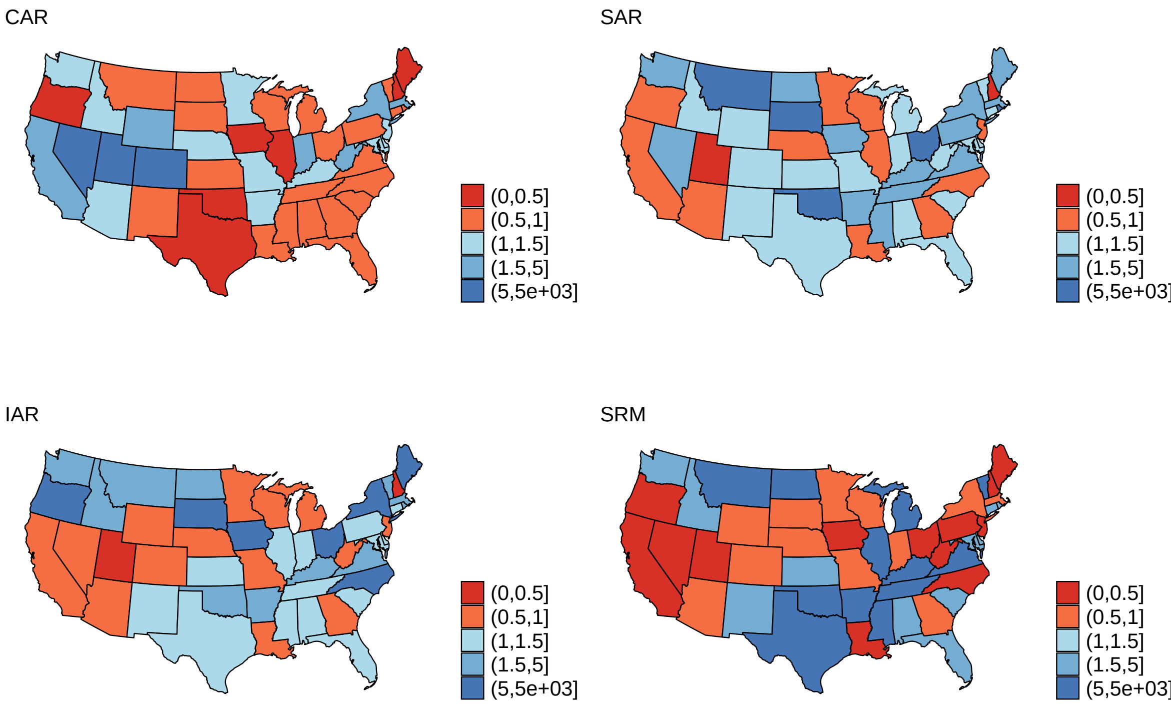

Figure 4.1: Squared deviations of the independent FH model over the four spatial models. The value larger (smaller) than one indicates a spatial model has a smaller (larger) squared deviation. The blue (red) color scheme represents states where a spatial model outperform (underperform) the FH model.

For each data set, we fit the independent FH model and four spatial models as specified in (16)–(18) with $\alpha = 0$ for (18). Then the squared deviations of unsampled areas are obtained for each model. Specifically, for $j = 1, \ldots, m_1$, we calculate $r_{kj}^2 = (\hat{\theta}_{kj} - \theta_j)^2$, where $\hat{\theta}_{kj}$ is the posterior mean of θ_j under the kth model with the missing direct estimates $Y_1, \ldots, Y_{m_1}$. Each panel displayed in Figure 4.1 illustrates the following quantities:

$$\xi_{ki} = \frac{r_{1i}^2}{r_{ki}^2}, \quad i = 1, \ldots, m, \tag{22}$$

where $k = 2, \ldots, 5$. A value of ξ_{ki} less than 1 favors the independent FH model and we identify this by depicting the state in red color scheme in the map. On the other hand, a value of ξ_{ki} larger than 1 indicates that the kth, $k = 2, \ldots, 5$, model has a smaller squared deviation on the ith small area. The darker the red (blue) color is the smaller (larger) ξ_{ki} is.

The top left panel of Figure 4.1 indicates that only for 21 states the CAR model produces squared deviations that are smaller than that of the independent FH model, and for 28 states the FH model squared deviations are smaller. The FH independent model is better. Of the 28 states with smaller FH squared deviations, seven of them have squared deviations less than half of the squared deviations of the CAR model, and remaining twenty-one states have smaller squared deviations but more than half the corresponding value from the CAR model.

The results with the SAR model are illustrated at the top right panel. The SAR model has smaller squared deviations for 36 out of 49 states, and for only 13 states smaller squared deviations for the FH model. Among the 36 states with larger FH squared deviations, five states have at least five times bigger, fourteen states have between 1.5 and 5 times bigger, and seventeen states have between 1 and 1.5 times bigger.

The IAR model outperforms the independent FH model in 33 states. Among the 33 states with larger FH squared deviations, seven states have at least five times bigger, eleven of them have between 1.5 and 5 times bigger, and the remaining fifteen states have between 1 and 1.5 times bigger. Overall, it performs similarly to the SAR model but the results are more volatile than the SAR model.

Lastly, the bottom right panel presents results for the SRM model. The SRM model outperforms the independent FH model in 24 states. Of the 24 states with larger FH squared deviations, 12 states have at least five times bigger, and the remaining twelve of them have between 1.5 and 5 times bigger.

5 Conclusions

In this paper, we followed a Bayesian approach to investigate four popular spatial random effects models as alternatives to the independent Fay-Herriot model to estimate small area means. In particular, we considered four spatial models with different autocorrelation structures. We further extended the spatial models to allow multiple small areas without any direct estimates in predicting small area means for all the areas. For a class of improper noninformative priors, we established posterior proprieties of the proposed models for both setups.

A simulation study in Section 3 showed that prediction accuracy can be greatly improved by considering spatial models when effective covariates are unavailable. Datta et al. (2011) has noted that the prediction accuracy of small area estimation models largely depends on the availability of good covariates. In other words, when good covariates are unavailable, we may not expect much improvement from the independent

Fay-Herriot model over direct estimates. The simulation results indicated that in such cases, the spatial models significantly improved the prediction accuracy by exploiting information from adjacent areas.

We applied various spatial random effects models to estimate four-person family median incomes of 49 U.S. contiguous states. Even when a good covariate exists, the spatial models exhibited noticeable improvement in terms of average squared deviations and average posterior standard deviations. When a good covariate was unavailable or unused, spatial models provided significantly more accurate median income predictions (measured by the average squared deviation) with much smaller variability (measured by the average posterior standard deviation). This conclusion was consistent with the simulation results when a more effective covariate was not included in the regression model. Furthermore, the SAR and IAR models provided more precise small area estimates when some small areas do not have direct estimates. In summary, the spatial models considered in this paper outperform the independent Fay-Herriot model. A significant improvement over the independent Fay-Herriot model can be expected when effective covariates are unavailable. Since useful covariates are not always available, the utility of the proposed method based on spatial models in small area estimation can be substantial. Finally, our Bayesian approach straightforwardly provided the point estimate and measure of uncertainty of that estimate of a small area mean.

6 Proof of Propriety of Posterior Probability Densities

Proof of Theorem 1. For convenience of notation, we denote $\mathbf{\Omega}_k(\rho)$ by $\mathbf{\Omega}_k$, and, for a given square matrix M, the determinant of M is denoted by $|M|$. Let $\delta = \max_i D_i < \infty$. Then, for a generic positive finite constant K,

$$f(\mathbf{y}, \boldsymbol{\theta}, \boldsymbol{\beta}, \sigma_v^2, \rho) \leq K g(\sigma_v^2) h(\rho) \exp\left\{-\frac{1}{2\delta}\sum_{i=1}^{m}(y_i - \theta_i)^2\right\} \tag{23}$$

$$\times |\sigma_v^2 \mathbf{\Omega}_k^{-1}|^{-1/2} \exp\left\{-\frac{1}{2}(\boldsymbol{\theta} - \mathbf{X}\boldsymbol{\beta})^\top (\sigma_v^2 \mathbf{\Omega}_k^{-1})^{-1} (\boldsymbol{\theta} - \mathbf{X}\boldsymbol{\beta})\right\}. \tag{24}$$

Integrating both sides with respect to $\boldsymbol{\theta}$, we get

$$\begin{aligned}
&\int f(\mathbf{y},\boldsymbol{\theta},\boldsymbol{\beta},\sigma_v^2,\rho)d\boldsymbol{\theta}\\
&\qquad \leq K g(\sigma_v^2)h(\rho)|\delta\mathbf{I}_m+\sigma_v^2\boldsymbol{\Omega}_k^{-1}|^{-1/2}\exp\left\{-\frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^\top(\delta\mathbf{I}_m+\sigma_v^2\boldsymbol{\Omega}_k^{-1})^{-1}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\right\}.
\end{aligned}$$

We now derive upper bounds for

$$|\delta\mathbf{I}_m+\sigma_v^2\boldsymbol{\Omega}_k^{-1}|^{-1/2}\exp\left\{-\frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^\top(\delta\mathbf{I}_m+\sigma_v^2\boldsymbol{\Omega}_k^{-1})^{-1}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\right\},$$

for $k=2,\ldots,5$.

Details for the CAR Model

We first consider the CAR model where $k=2$. Let P_W be an orthogonal matrix such that $\mathbf{P}_\mathbf{W}^\top\mathbf{W}\mathbf{P}_\mathbf{W}=\mathrm{diag}\{\lambda_i\}_{i=1}^m=\boldsymbol{\Lambda}$. Then $\boldsymbol{\Omega}_2(\rho)^{-1}=\mathbf{P}_\mathbf{W}\{I-\rho\boldsymbol{\Lambda}\}^{-1}\mathbf{P}_\mathbf{W}^\top$ and hence

$$\begin{aligned}
(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^\top&(\delta\mathbf{I}_m+\sigma_v^2\boldsymbol{\Omega}_2^{-1})^{-1}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\\
&=(\mathbf{P}_\mathbf{W}^\top\mathbf{y}-\mathbf{P}_W^\top\mathbf{X}\boldsymbol{\beta})^\top(\delta\mathbf{I}_m+\sigma_v^2\{\mathbf{I}-\rho\boldsymbol{\Lambda}\}^{-1})^{-1}(\mathbf{P}_W^\top\mathbf{y}-\mathbf{P}_\mathbf{W}^\top\mathbf{X}\boldsymbol{\beta})\\
&=(\mathbf{y}_*-\mathbf{X}_*\boldsymbol{\beta})^\top(\delta\mathbf{I}_m+\sigma_v^2\{\mathbf{I}-\rho\boldsymbol{\Lambda}\}^{-1})^{-1}(\mathbf{y}_*-\mathbf{X}_*\boldsymbol{\beta}),
\end{aligned}$$

where $\mathbf{y}_*=\mathbf{P}_\mathbf{W}^\top\mathbf{y}$ and $\mathbf{X}_*=\mathbf{P}_\mathbf{W}^\top\mathbf{X}$. Suppose the rows of $\mathbf{X}_*$ corresponding to distinct p indices $\{i_1,\ldots,i_p\}\subseteq\{1,\ldots,m\}$ are linearly independent. We denote these rows by $\mathbf{x}_{i_k*}^\top$, $k=1,\ldots,p$. Define the $p\times p$ non-singular matrix $[\mathbf{x}_{i_1*},\ldots,\mathbf{x}_{i_p*}]^\top$ by $\mathbf{A}$. Also, let $\boldsymbol{\eta}=(\eta_1,\ldots,\eta_p)^\top=\mathbf{A}\boldsymbol{\beta}$. Note that

$$(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^\top(\delta\mathbf{I}_m+\sigma_v^2\boldsymbol{\Omega}_2^{-1})^{-1}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\geq\sum_{k=1}^p\frac{(y_{i_k*}-\eta_{i_k})^2}{\delta+\sigma_v^2(1-\rho\lambda_{i_k})^{-1}}.$$

From this, we get that

$$\begin{aligned}
\int\exp\left\{-\frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^\top(\delta\mathbf{I}_m+\sigma_v^2\boldsymbol{\Omega}_2^{-1})^{-1}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\right\}d\boldsymbol{\beta}&\leq\int\exp\left\{-\frac{1}{2}\sum_{k=1}^p\frac{(y_{i_k*}-\eta_{i_k})^2}{\delta+\sigma_v^2(1-\rho\lambda_{i_k})^{-1}}\right\}d\boldsymbol{\beta}\\
=\int\exp\left\{-\frac{1}{2}\sum_{k=1}^p\frac{(y_{i_k*}-\eta_{i_k})^2}{\delta+\sigma_v^2(1-\rho\lambda_{i_k})^{-1}}\right\}d\boldsymbol{\eta}|\mathbf{A}^\top\mathbf{A}|^{-1/2}&\\
=K\prod_{k=1}^p\{\,\delta+\sigma_v^2(1-\rho\lambda_{i_k})^{-1}\}^{1/2}, \qquad (25)&
\end{aligned}$$

where $K > 0$ is a finite generic constant. Also, we know that

$$|\delta \mathbf{I}_m + \sigma_v^2 \boldsymbol{\Omega}_2^{-1}|^{-1/2} = \prod_{i=1}^{m} \left\{ \delta + \sigma_v^2 (1-\rho\lambda_i)^{-1} \right\}^{-1/2}. \tag{26}$$

By (25) and (26), we get

$$\begin{aligned}
|\delta \mathbf{I}_m + \sigma_v^2 \boldsymbol{\Omega}_2^{-1}|^{-1/2} & \int \exp\left\{ -\frac{1}{2} (\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^\top (\delta \mathbf{I}_m + \sigma_v^2 \boldsymbol{\Omega}_2^{-1})^{-1} (\mathbf{y}-\mathbf{X}\boldsymbol{\beta}) \right\} d\boldsymbol{\beta} \\
&\le K \prod_{i \notin \{i_1,\ldots,i_p\}} \left\{ \delta + \sigma_v^2 (1-\rho\lambda_i)^{-1} \right\}^{-1/2} \\
&\le K \left\{ 1(\sigma_v^2 < N) + (\sigma_v^2)^{-(m-p)/2} \prod_{i \notin \{i_1,\ldots,i_p\}} (1-\rho\lambda_i)^{1/2} 1(\sigma_v^2 > N) \right\}
\end{aligned} \tag{27}$$

for any positive number N. Recall that $\lambda_m^{-1} < \rho < \lambda_1^{-1}$. We know $1-\rho\lambda_i$ is an eigenvalue of $\boldsymbol{\Omega}_2$. Thus, for $\lambda_m^{-1} < \rho < \lambda_1^{-1}$, for $i = 1, \cdots, m$, $1-\rho\lambda_i > 0$. Also, $\sum_{i=1}^m (1-\rho\lambda_i) = m$. These imply that $0 < 1-\rho\lambda_i < m$. Then from (27), we get

$$\begin{aligned}
|\delta \mathbf{I}_m + \sigma_v^2 \boldsymbol{\Omega}_2^{-1}|^{-1/2} & \int \exp\left\{ -\frac{1}{2} (\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^\top (\delta \mathbf{I}_m + \sigma_v^2 \boldsymbol{\Omega}_2^{-1})^{-1} (\mathbf{y}-\mathbf{X}\boldsymbol{\beta}) \right\} d\boldsymbol{\beta} \\
&\le K \left\{ 1(\sigma_v^2 < N) + (\sigma_v^2)^{-(m-p)/2} 1(\sigma_v^2 > N) \right\}.
\end{aligned} \tag{28}$$

From (28), propriety of the posterior will follow under the conditions of the theorem.

Details for the SAR Model

We now consider $k = 3$ for the SAR model. With $\mathbf{W}_* = \mathbf{L}^{-1/2} \mathbf{W} \mathbf{L}^{-1/2}$, we have

$$\begin{aligned}
\boldsymbol{\Omega}_3 &= (\mathbf{I}_m - \rho \widetilde{\mathbf{W}})^\top (\mathbf{I}_m - \rho \widetilde{\mathbf{W}}) \\
&= (\mathbf{L} - \rho \mathbf{W})^\top \mathbf{L}^{-2} (\mathbf{L} - \rho \mathbf{W}) \\
&= \mathbf{L}^{1/2} (\mathbf{I}_m - \rho \mathbf{W}_*) \mathbf{L}^{-1} (\mathbf{I}_m - \rho \mathbf{W}_*) \mathbf{L}^{1/2}.
\end{aligned}$$

First, $\mathrm{tr}\, \boldsymbol{\Omega}_3 = m + \rho^2 \sum_i \sum_j \tilde{w}_{ij}^2 \le m + \rho^2 \sum_i \sum_j \tilde{w}_{ij} = m + \rho^2 m < 2m$ since $0 \le \tilde{w}_{ij} \le 1$, $\sum_j \tilde{w}_{ij} = 1$, and $-1 < \rho < 1$.

Note that the eigenvalues $\nu_1, \ldots, \nu_m$ of $\mathbf{W}_*$ are all real (since $\mathbf{W}_*$ is symmetric). Also, $\mathbf{W}_*$ and $\widetilde{\mathbf{W}}$ have identical eigenvalues. Being a stochastic matrix, $\widetilde{\mathbf{W}}$ has at least one eigenvalue equal to one and the remaining eigenvalues are bounded by 1, that is

$|\nu_i| \leq 1$ and $\max_i \nu_i = 1$. For $-1 < \rho < 1$, $1-\rho\nu_i > 0$. Then, $|\mathbf{\Omega}_3| = \prod_{i=1}^m (1-\rho\nu_i)^2 > 0$. Thus, the eigenvalues of $\mathbf{\Omega}_3$ are positive and bounded above by $2m$. Let $\ell_{(1)} = \min \ell_i$ and $\ell_{(m)} = \max \ell_i$, where $L = \mathrm{diag}\{\ell_i\}_{i=1}^m$. Then $\ell_{(1)} > 0$ and $\ell_{(m)}$ is bounded above. By writing

$$\mathbf{\Sigma}_3 = \delta \mathbf{I}_m + \sigma_v^2 \mathbf{\Omega}_3^{-1} = \mathbf{L}^{-1/2}\{\delta \mathbf{L} + \sigma_v^2 (\mathbf{I}_m - \rho \mathbf{W}_*)^{-1} \mathbf{L} (\mathbf{I}_m - \rho \mathbf{W}_*)^{-1}\} \mathbf{L}^{-1/2},$$

we have

$$\begin{aligned} |\mathbf{\Sigma}_3| &= |\mathbf{L}|^{-1} |\delta \mathbf{L} + \sigma_v^2 (\mathbf{I}_m - \rho \mathbf{W}_*)^{-1} \mathbf{L} (\mathbf{I}_m - \rho \mathbf{W}_*)^{-1}| \\ &\geq |\mathbf{L}|^{-1} \ell_{(1)}^m \ |\delta \mathbf{I}_m + \sigma_v^2 (\mathbf{I}_m - \rho \mathbf{W}_*)^{-2}| \\ &= |\mathbf{L}|^{-1} \ell_{(1)}^m \ \prod_{i=1}^m \left\{\delta + \sigma_v^2 (1-\rho\nu_i)^{-2}\right\}, \end{aligned} \tag{29}$$

We also note that

$$\begin{aligned} &(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^\top \mathbf{\Sigma}_3^{-1} (\mathbf{y} - \mathbf{X}\boldsymbol{\beta}) \\ &= (\mathbf{L}^{1/2}\mathbf{y} - \mathbf{L}^{1/2}\mathbf{X}\boldsymbol{\beta})^\top \left\{\delta \mathbf{L} + \sigma_v^2 (\mathbf{I}_m - \rho \mathbf{W}_*)^{-1} \mathbf{L} (\mathbf{I}_m - \rho \mathbf{W}_*)^{-1}\right\}^{-1} (\mathbf{L}^{1/2}\mathbf{y} - \mathbf{L}^{1/2}\mathbf{X}\boldsymbol{\beta}) \\ &= (\mathbf{r} - \mathbf{S}\boldsymbol{\beta})^\top \left\{\delta \mathbf{L} + \sigma_v^2 (\mathbf{I}_m - \rho \mathbf{W}_*)^{-1} \mathbf{L} (\mathbf{I}_m - \rho \mathbf{W}_*)^{-1}\right\}^{-1} (\mathbf{r} - \mathbf{S}\boldsymbol{\beta}) \\ &\geq (\ell_{(m)}^{-1/2}\mathbf{r} - \ell_{(m)}^{-1/2}\mathbf{S}\boldsymbol{\beta})^\top \left\{\delta \mathbf{I}_m + \sigma_v^2 (\mathbf{I}_m - \rho W_*)^{-2}\right\}^{-1} (\ell_{(m)}^{-1/2}\mathbf{r} - \ell_{(m)}^{-1/2}\mathbf{S}\boldsymbol{\beta}) \\ &= (\tilde{\mathbf{r}} - \tilde{\mathbf{S}}\boldsymbol{\beta})^\top \left\{\delta \mathbf{I}_m + \sigma_v^2 (\mathbf{I}_m - \rho \mathbf{M}_*)^{-2}\right\}^{-1} (\tilde{\mathbf{r}} - \tilde{\mathbf{S}}\boldsymbol{\beta}) \\ &\geq \sum_{k=1}^p \frac{(\tilde{r}_{i_k} - \tilde{\mathbf{s}}_{i_k}^\top \boldsymbol{\beta})^2}{\delta + \sigma_v^2 (1-\rho\nu_{i_k})^{-2}}, \end{aligned} \tag{30}$$

where $\mathbf{r} = \mathbf{L}^{1/2}\mathbf{y}$, $\mathbf{S} = \mathbf{L}^{1/2}\mathbf{X}$, $\tilde{\mathbf{r}} = \ell_{(m)}^{-1/2} \mathbf{P}_{\mathbf{W}_*} \mathbf{r}$, $\tilde{\mathbf{S}} = \ell_{(m)}^{-1/2} \mathbf{P}_{\mathbf{W}_*} \mathbf{S}$, $\mathbf{M}_* = \mathrm{diag}\{\nu_i\}_{i=1}^m$, $\mathbf{P}_{\mathbf{W}_*}^\top$ is an orthogonal matrix of eigenvectors of $\mathbf{W}_*$, and $\{i_1, \ldots, i_p\}$ is a subset of $\{1, \ldots, m\}$ so that the $p \times p$ matrix $[\tilde{\mathbf{s}}_{i_1}, \ldots, \tilde{\mathbf{s}}_{i_p}]^\top = \tilde{\mathbf{S}}_1$, a submatrix of $\tilde{\mathbf{S}}$, is non-singular. Note that $\tilde{\mathbf{S}}_1$ is determined by $\mathbf{W}$. Using (30) we get

$$\int \exp\{-\frac{1}{2}(\mathbf{y} - \mathbf{X}\boldsymbol{\beta})^\top \mathbf{\Sigma}_3^{-1} (\mathbf{y} - \mathbf{X}\boldsymbol{\beta})\} d\boldsymbol{\beta} \leq (2\pi)^{p/2} |\tilde{\mathbf{S}}_1^\top \tilde{\mathbf{S}}_1|^{-1/2} \prod_{k=1}^p \left\{\delta + \sigma_v^2 (1-\rho\nu_{i_k})^{-2}\right\}^{1/2}. \tag{31}$$

Based on (29) and (31), we get that

$$\begin{aligned}
\int |\boldsymbol{\Sigma}_3|^{-1/2} \exp\{ -\frac{1}{2}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^\top \boldsymbol{\Sigma}_3^{-1}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})\} d\boldsymbol{\beta} & \\
&\leq K \prod_{i \notin \{i_1,\ldots,i_p\}} \left\{\delta + \sigma_v^2(1-\rho\nu_i)^{-2}\right\}^{-1/2} \\
&\leq K \left\{ 1(\sigma_v^2 < N) + (\sigma_v^2)^{-(m-p)/2} 1(\sigma_v^2 > N) \prod_{i \notin \{i_1,\ldots,i_p\}} (1-\rho\nu_i) \right\} \\
&\leq K \left\{ 1(\sigma_v^2 < N) + (\sigma_v^2)^{-(m-p)/2} 1(\sigma_v^2 > N) \right\}, \qquad (32)
\end{aligned}$$

where we use the fact that $-1 < \rho < 1$ and $-1 \leq \nu_i \leq 1$ to claim $0 < 1 - \rho\nu_i < 2$. Again, proceeding along that lines we did for the CAR model, we can establish from (32), the propriety of the posterior pdf under the conditions stated in the theorem.

Details for the IAR Model

We now consider $k = 4$ for the IAR model where

$$\boldsymbol{\Omega}_4 = \mathbf{L} - \rho\mathbf{W} = \mathbf{L}^{1/2}(\mathbf{I}_m - \rho\mathbf{W}_*)\mathbf{L}^{1/2}.$$

Let $\boldsymbol{\Sigma}_4 = \delta\mathbf{I}_m + \sigma_v^2\boldsymbol{\Omega}_4^{-1} = \mathbf{L}^{-1/2}\{\delta\mathbf{L} + \sigma_v^2(\mathbf{I}_m - \rho\mathbf{W}_*)^{-1}\}\mathbf{L}^{-1/2}$. Then

$$|\boldsymbol{\Sigma}_4| \geq |\mathbf{L}|^{-1} k_*^m \prod_{i=1}^{m} \left\{\delta + \sigma_v^2(1-\rho\nu_i)^{-1}\right\}, \qquad (33)$$

where $k_* = \min\{\ell_{(1)}, 1\}$. Proceeding along the same line as in (30), we get that

$$(\mathbf{y}-\mathbf{X}\boldsymbol{\beta})^\top \boldsymbol{\Sigma}_4^{-1}(\mathbf{y}-\mathbf{X}\boldsymbol{\beta}) \geq \sum_{k=1}^{p} \frac{(\tilde{r}_{i_k} - \tilde{\mathbf{s}}_{i_k}^\top\boldsymbol{\beta})^2}{\delta + \sigma_v^2(1-\rho\nu_{i_k})^{-1}}. \qquad (34)$$

Again, as we had for the two previous cases, we can use (33) and (34) to establish the propriety of the posterior pdf under the conditions stated in the theorem.

Details for the SRM Model

Finally, we consider $k = 5$, where for the SRM case we have

$$\boldsymbol{\Omega}_5 = \rho\mathbf{R} + (1-\rho)\mathbf{I}_m.$$

Suppose $r_1, \ldots, r_m$ are the eigenvalues of $\mathbf{R}$ and $\mathbf{P_R}$ is an orthogonal matrix such that $\mathbf{P}_{\mathbf{R}}^{\top}\mathbf{R}\mathbf{P}_{\mathbf{R}} = \mathrm{diag}\{r_i\}_{i=1}^m$. Since $\mathbf{R}$ is a non-negative definite matrix, $r_i \geq 0$, $i = 1, \ldots, m$, and $\sum_{i=1}^m r_i = \mathrm{tr}\,\mathbf{R} = \sum_{i=1}^m \ell_i$, implying that $r_1, \ldots, r_m$ are all bounded between 0 and $\ell = \sum_{i=1}^m \ell_i$. Then we can write

$$\mathbf{\Omega}_5 = \mathbf{P_R}\{\mathrm{diag}(\rho r_i + 1 - \rho)_{i=1}^m\}\mathbf{P}_{\mathbf{R}}^{\top},$$

and claim that for $0 < \rho < 1$, the eigenvalues of $\mathbf{\Omega}_5$ are all positive and bounded above by $\sum_{i=1}^m r_i + 1 = \ell + 1$. Then, with $\tilde{\mathbf{r}} = \mathbf{P}_{\mathbf{R}}^{\top}\mathbf{y}$, and $\tilde{\mathbf{S}} = \mathbf{P}_{\mathbf{R}}^{\top}\mathbf{X}$, we can establish an inequality similar to (30). Note that the nonsingular matrix $\tilde{\mathbf{S}}_1$ is a submatrix of $\tilde{\mathbf{S}}$ and is free from ρ. Boundedness of the eigenvalues of $\mathbf{\Omega}_5$ will lead to an inequality similar to (33). Finally, the propriety will be attained under the conditions in the theorem.

$\square$

Proof of Corollary 1.1 . The result is directly obtained by Theorem 1 as follows. Since $h(\rho) = 1(\ell_k < \rho < u_k)$ is integrable, it suffices to show the integrability with $g(\sigma_v^2) = (\sigma_v^2)^{-\alpha}$. We know that

$$\int_0^N (\sigma_v^2)^{-\alpha} d\sigma_v^2 < \infty \quad \text{and} \quad \int_N^\infty (\sigma_v^2)^{-(2\alpha+m-p)/2} d\sigma_v^2 < \infty,$$

if $\alpha < 1$ and $(2\alpha + m - p)/2 > 1$, respectively. Thus the posterior pdf will be proper if $1 - (m-p)/2 < \alpha < 1$. $\square$

Proof of Theorem 2. Let $m_1 \geq 0$ be the number of small areas with no direct estimates and let $m_2 = m - m_1$. Also, let $\mathbf{Y}_{(2)}$ be the $m_2 \times 1$ vector with direct estimates corresponding to the sampled small areas. Without loss of generality, we assume that $\theta_1, \ldots, \theta_m$ are arranged so that $\boldsymbol{\theta} = (\boldsymbol{\theta}_{(1)}^{\top}, \boldsymbol{\theta}_{(2)}^{\top})^{\top}$. Let $\mathbf{D}_{(2)} = \{D_i\}_{i=m_1+1}^m$ be the diagonal matrix with sampling variances corresponding to the components of $\mathbf{Y}_{(2)}$ and $\delta = \max_{m_1 < i \leq m} D_i < \infty$. For convenience of notation, we denote $\mathbf{\Omega}_k(\rho)$ by $\mathbf{\Omega}$, and as before, any generic positive finite constant will be denoted by K.

The joint pdf of $\mathbf{Y}_{(2)}, \boldsymbol{\theta}, \boldsymbol{\beta}, \sigma_v^2$ and ρ is given by

$$f(\mathbf{y}_{(2)}, \boldsymbol{\theta}, \boldsymbol{\beta}, \sigma_v^2, \rho) = \mathcal{N}_{m_2}(\mathbf{y}_{(2)}|\boldsymbol{\theta}_{(2)}, \mathbf{D}_{(2)})\mathcal{N}_m(\boldsymbol{\theta}|\mathbf{X}\boldsymbol{\beta}, \sigma_v^2\mathbf{\Omega}^{-1})g(\sigma_v^2)h(\rho), \tag{35}$$

where $\mathcal{N}_{m_2}(\mathbf{y}_{(2)}|\boldsymbol{\theta}_{(2)}, \mathbf{D}_{(2)})$ is the multivariate normal pdf with the mean $\boldsymbol{\theta}_{(2)}$ and covariance matrix $\mathbf{D}_{(2)}$. Since

$$\mathcal{N}_{m_2}(\mathbf{y}_{(2)}|\boldsymbol{\theta}_{(2)}, \mathbf{D}_{(2)}) \leq K \exp\left\{-\frac{1}{2\delta}(\mathbf{y}_{(2)} - \boldsymbol{\theta}_{(2)})^\top(\mathbf{y}_{(2)} - \boldsymbol{\theta}_{(2)})\right\},$$

we have

$$\begin{aligned}\pi(\boldsymbol{\theta}, \boldsymbol{\beta}, \sigma_v^2, \rho|\mathbf{y}_{(2)}) &\leq K \exp\left\{-\frac{1}{2\delta}(\mathbf{y}_{(2)} - \boldsymbol{\theta}_{(2)})^\top(\mathbf{y}_{(2)} - \boldsymbol{\theta}_{(2)})\right\} \mathcal{N}_m(\boldsymbol{\theta}|\mathbf{X}\boldsymbol{\beta}, \sigma_v^2\boldsymbol{\Omega}^{-1})g(\sigma_v^2)h(\rho) \\ &= K \int \exp\left\{-\frac{1}{2\delta}(\mathbf{y} - \boldsymbol{\theta})^\top(\mathbf{y} - \boldsymbol{\theta})\right\} d\mathbf{y}_{(1)}\mathcal{N}_m(\boldsymbol{\theta}|\mathbf{X}\boldsymbol{\beta}, \sigma_v^2\boldsymbol{\Omega}^{-1})g(\sigma_v^2)h(\rho). \qquad (36)\end{aligned}$$

By integrating both sides of (36) with respect to θ, we get

$$\pi(\boldsymbol{\beta}, \sigma_v^2, \rho|\mathbf{y}_{(2)}) \leq K g(\sigma_v^2)h(\rho) \int \mathcal{N}_m(\mathbf{y}|\mathbf{X}\boldsymbol{\beta}, \delta\mathbf{I}_m + \sigma_v^2\boldsymbol{\Omega}^{-1})d\mathbf{y}_{(1)}. \qquad (37)$$

Partition $\mathbf{X}$ as $\mathbf{X} = [\mathbf{X}_1, \mathbf{X}_2]^\top$, where $\mathbf{X}_1^\top$ is $m_1 \times p$ and $\mathbf{X}_2^\top$ is $m_2 \times p$. We assume that $\text{rank}(\mathbf{X}_2) = p$. Let $\mathbf{d} = (0_{m_1}^\top, \mathbf{y}_{(2)}^\top)^\top$, $\boldsymbol{\phi} = (\mathbf{y}_{(1)}^\top, \boldsymbol{\beta}^\top)^\top$ and

$$\mathbf{G} = \begin{bmatrix} -\mathbf{I}_{m_1} & \mathbf{X}_1^\top \\ 0_{m_2,m_1} & \mathbf{X}_2^\top \end{bmatrix}.$$

Then, we can write

$$\mathbf{y} - \mathbf{X}\boldsymbol{\beta} = \mathbf{d} - \mathbf{G}\boldsymbol{\phi},$$

where $\mathbf{G}$ is $m \times (m_1 + p)$, $\boldsymbol{\phi}$ is $(m_1 + p) \times 1$. Hence, (37) can be written as

$$\pi(\boldsymbol{\beta}, \sigma_v^2, \rho|\mathbf{y}_{(2)}) \leq K g(\sigma_v^2)h(\rho) \int \mathcal{N}_m(\mathbf{d}|\mathbf{G}\boldsymbol{\phi}, \delta\mathbf{I}_m + \sigma_v^2\boldsymbol{\Omega}^{-1})d\mathbf{y}_{(1)}. \qquad (38)$$

By integrating both sides of (38) with respect to $\boldsymbol{\beta}$, we get

$$\pi(\sigma_v^2, \rho|\mathbf{y}_{(2)}) \leq K g(\sigma_v^2)h(\rho) \int \mathcal{N}_m(\mathbf{d}|\mathbf{G}\boldsymbol{\phi}, \delta\mathbf{I}_m + \sigma_v^2\boldsymbol{\Omega}^{-1})d\boldsymbol{\phi}. \qquad (39)$$

Since $\text{rank}(\mathbf{X}_2) = p$, we immediately get that $\text{rank}(\mathbf{G}) = m_1 + p$. Thus $\mathbf{G}$ has full column rank. Now if we take $\mathbf{G}$ in place of $\mathbf{X}$ and $\boldsymbol{\phi}$ in place of $\boldsymbol{\beta}$ in Theorem 1, and we proceed as in Theorem 1, the propriety of the posterior pdf will follow under the stated conditions of the Theorem.

□

A Supplementary figures

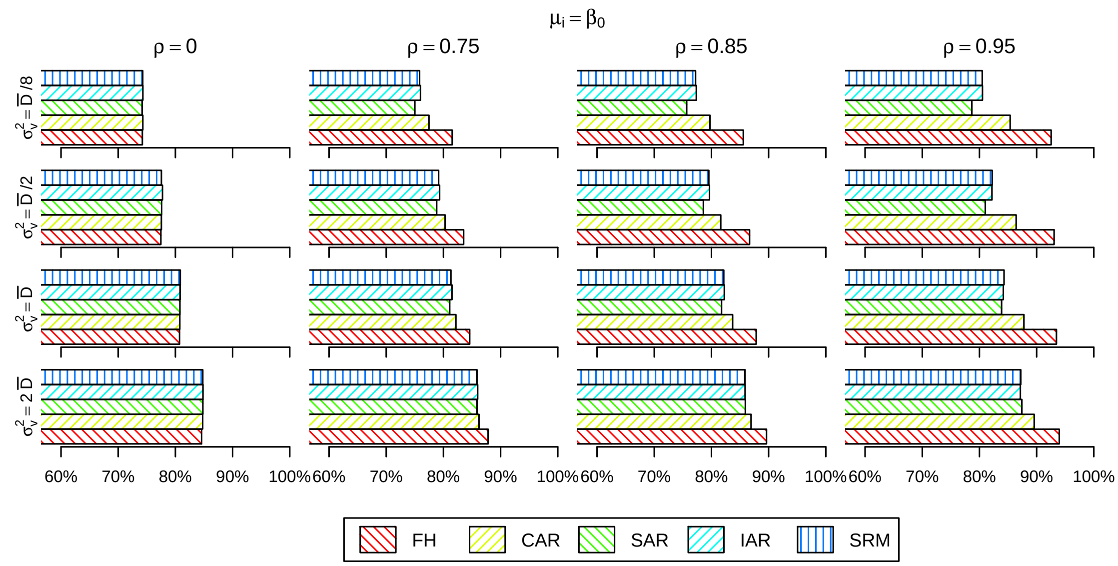

Figure A.1: Ratio of average posterior standard deviation to average sampling standard deviation when only the intercept is in the fitted model.

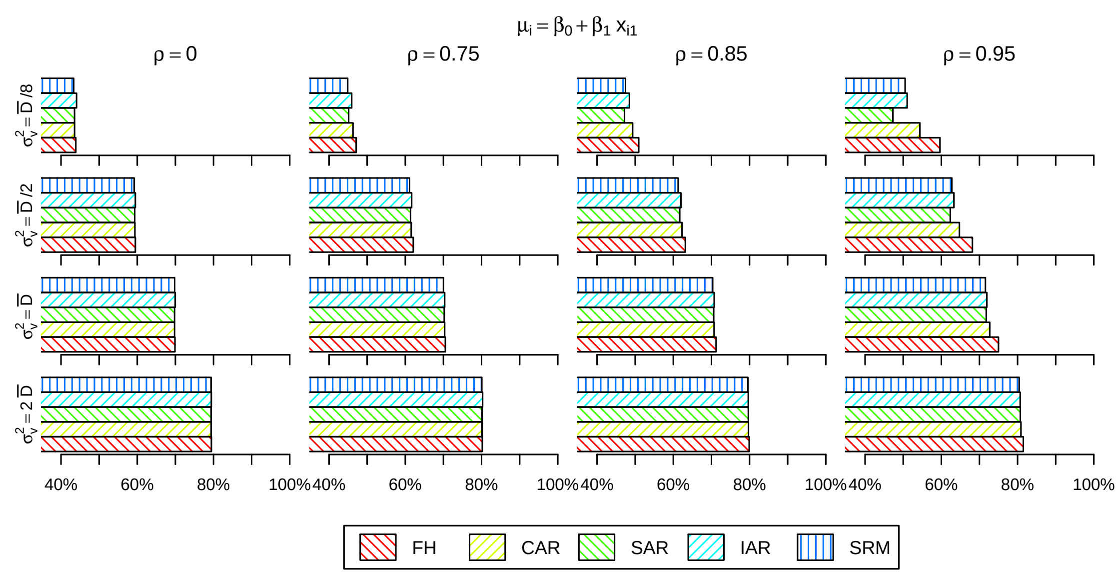

Figure A.2: Ratio of average posterior standard deviation to average sampling standard deviation when only the covariate x_1 is in the fitted model.

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