Unit Root Properties of Seasonal Adjustment and Related Filters: Special Cases

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Abstract

Bell (2012) catalogued unit root factors contained in linear filters used in seasonal adjustment (model-based or from the X-11 method) but noted that, for model-based seasonal adjustment, special cases could arise where filters could contain more unit root factors than was indicated by the general results. This note reviews some special cases that occur with canonical ARIMA model-based adjustment in which, with some commonly used ARIMA models, the symmetric seasonal filters contain two extra nonseasonal differences (i.e., they include an extra \((1 - B)(1 - F)\)). This increases by two the degree of polynomials in time that are annihilated by the seasonal filter and reproduced by the seasonal adjustment filter. Other results for canonical ARIMA adjustment that are reported in Bell (2012), including properties of the trend and irregular filters, and properties of the asymmetric and finite filters, are unaltered in these special cases. Special cases for seasonal adjustment with structural ARIMA component models are also briefly discussed.

Key Words: time series, linear filter, ARIMA model-based seasonal adjustment, canonical decomposition

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1 Introduction

The additive decomposition used for seasonal adjustment is

\[ y_t = S_t + T_t + I_t \]  \hspace{1cm} (1)

where \( y_t \) is the observed time series (possibly after transformation, e.g., taking logarithms), and \( S_t, T_t, \) and \( I_t \) are the seasonal, trend, and irregular components. We also let \( N_t = T_t + I_t = y_t - S_t \) denote the nonseasonal component, the estimate of which is known as the seasonally adjusted series. Many of the models proposed for model-based seasonal adjustment use component models that can be written in the following form:

\[ U(B)S_t = u_t(1 - B)^d \]

\[ (1 - B)^d T_t = v_t \]

\[ I_t \sim \text{i.i.d. } N(0, \sigma^2_I) \]  \hspace{1cm} (2)

where \( U(B) = 1 + B + \cdots + B^s \), \( B \) is the backshift operator \((BS_t = S_{t-1})\), \( s \) is the seasonal period, and \( u_t \) and \( v_t \) are independent of each other and of \( I_t \). Often \( u_t \) and \( v_t \) are assumed to follow stationary autoregressive-moving average models (Box and Jenkins 1970), in which case \( y_t \) follows an ARIMA (autoregressive-integrated-moving average) model that can be written

\[ \phi(B)(1 - B)^{d-1}(1 - B^s)y_t = \theta(B)a_t \]  \hspace{1cm} (3)

where \( \phi(B) = 1 - \phi_1 B - \cdots - \phi_p B^p \) is the AR operator, \( \theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q \) is the MA operator, and \( a_t \) is white noise, \( i.i.d. \ N(0, \sigma^2_a) \). The operators \( \phi(B) \) and \( \theta(B) \), which may be products of nonseasonal and seasonal polynomials in \( B \), are assumed to have all their zeros outside the unit circle. The expression of the model as in (3) requires \( d \geq 1 \), which is standard in seasonal adjustment practice. Note that \( 1 - B^s = (1 - B)U(B) \) so \( (1 - B)^{d-1}(1 - B^s) = (1 - B)^d U(B) \).

This model framework covers the ARIMA model-based approach to seasonal adjustment as developed in Hillmer and Tiao (1982) and Burman (1980), and implemented in the TRAMO-SEATS
software of Gomez and Maravall (1997) and in the X-13-ARIMA-SEATS program (Monsell 2007). It also covers the structural components models of Harvey (1989), Durbin and Koopman (2001), and Kitagawa and Gersch (1984). Though Harvey did not formulate all his component models in ARIMA form, they can generally be written this way – see Bell (2004).

Let \( w_t = (1 - B)^d U(B) y_t \) be the differenced observed series. From (1),

\[
 w_t = (1 - B)^d u_t + U(B) v_t + (1 - B)^d U(B) I_t. 
\]  

(4)

Let \( \gamma_w(k) = \text{Cov}(w_t, w_{t+k}) \) and let \( \gamma_w(B) \) be the autocovariance generating function (ACGF) of \( w_t \), defined as \( \gamma_w(B) \equiv \sum_{k=-\infty}^{\infty} \gamma_w(k) B^k \), where we treat \( B \) for this purpose as a complex variable. Given the ARMA model \( \phi(B) w_t = \theta(B) a_t \), and the orthogonality of the components in (4), it follows that (Box and Jenkins 1970, p. 49)

\[
 \gamma_w(B) = \sigma_a^2 \theta(B) \theta(F) / \phi(B) \phi(F) \\
\quad = (1 - B)^d(1 - F)^d \gamma_u(B) + U(B) U(F) \gamma_v(B) + (1 - B)^d(1 - F)^d U(B) U(F) \sigma_I^2, 
\]  

(5)

where \( F = B^{-1} \). Given ARMA models for \( u_t \) and \( v_t \), analogous expressions to (5) can be given for their ACGFs, \( \gamma_u(B) \) and \( \gamma_v(B) \). From \( w_t = (1 - B)^d U(B) y_t \), the pseudo ACGF of \( y_t \) is defined as \( \gamma_y(B) = \gamma_w(B) / (1 - B)^d(1 - F)^d U(B) U(F) \). We also define \( z_t = (1 - B)^d N_t = v_t + (1 - B)^d I_t \) with ACGF \( \gamma_z(B) = \gamma_v(B) + (1 - B)^d(1 - F)^d \sigma_I^2 \).

Bell (2012, p. 445) notes that the minimum mean squared error (MMSE) linear signal extraction estimate of \( S_t \) given the full doubly infinite realization of the series \( \{y_t\} \) is (now letting \( F = B^{-1} \) denote the forward shift operator)

\[
 \hat{S}_t = \omega_S(B) y_t \quad \text{where} \quad \omega_S(B) = \frac{\gamma_u(B)}{\gamma_w(B)} (1 - B)^d(1 - F)^d. 
\]  

(7)
Analogous to (7), the linear filters for the MMSE estimates of $N_t$, $T_t$, and $I_t$ are

$$
\omega_N(B) = \frac{\gamma_z(B)}{\gamma_w(B)} U(B)U(F)
$$

(8)

$$
\omega_T(B) = \frac{\gamma_w(B)}{\gamma_w(B)} U(B)U(F)
$$

(9)

$$
\omega_I(B) = \frac{\sigma^2}{\gamma_w(B)} U(B)U(F)(1 - B)^d(1 - F)^d.
$$

(10)

Note also that since $\hat{N}_t = y_t - \hat{S}_t$ and $\hat{T}_t = \hat{N}_t - \hat{I}_t$, it follows that $\omega_N(B) = 1 - \omega_S(B)$ and $\omega_T(B) = 1 - \omega_S(B) - \omega_I(B)$.

Simple inspection of (7)–(10) led to results reported in Bell (2012) for unit root factors contained in these filters. There (Bell 2012, pp. 446-447) it was noted that:

Something not clear from [(7)–(10)] is whether these filters contain additional unit root factors beyond those obvious from inspection. Bell (2010) notes that $\omega_I(B)$ will not include additional unit root factors, while for $\omega_S(B)$, $\omega_N(B)$, and $\omega_T(B)$, additional unit root factors are possible if they appear in the MA polynomials of the ARIMA models for $S_t$, $N_t$, or $T_t$. For example, Hillmer and Tiao (1982, p. 67) examine a model for which the canonical trend component has a factor of $(1 + B)$ in its MA polynomial. While potential additional unit root factors in the filters considered can obviously be examined for any particular model, general results are difficult to give.

The polynomial factors in the MA operator of any ARMA model, such as $\theta(B)$ in (3), correspond to double factors in the numerator of the autocovariance generating function – note $\theta(B)\theta(F)$ in equation (5). So $1 - B$ is a factor of the MA polynomial of the model for $u_t$ if and only if the numerator of $\gamma_u(B)$ contains $(1 - B)(1 - F)$.

This note focuses on special cases that occur with canonical ARIMA model-based seasonal adjustment where, for some commonly used ARIMA models, and depending on the seasonal period $s$ and on the model parameter values, $\gamma_u(B)$ contains a factor of $(1 - B)(1 - F)$, which then provides an extra $(1 - B)(1 - F)$ in the symmetric seasonal filter $\omega_S(B)$. The explicit presence of
(1 − B)^d(1 − F)^d in the expression for ω_S(B) in (7) guarantees that ω_S(B) will always annihilate, and ω_N(B) = 1 − ω_S(B) will always reproduce, polynomials in t up to degree 2d − 1. When γ_a(B) contains (1 − B)(1 − F), this extra factor in ω_S(B) implies that it will annihilate, and ω_N(B) will reproduce, polynomials in t up to degree 2d + 1, which is two degrees higher than would otherwise be the case. For the common cases of d = 1 or 2, the extra (1 − B)(1 − F) means that the seasonal adjustment filter will reproduce cubic and quintic polynomials, respectively, instead of just linear and cubic polynomials. This property will not be shared by the corresponding trend filter ω_T(B) = 1 − ω_S(B) − ω_I(B) because, as noted in the quotation above, the corresponding canonical irregular filter will not include the extra (1 − B)(1 − F) factor.

Sections 2 and 3 provide results showing when the extra (1 − B)(1 − F) factor occurs in two models considered explicitly by Hillmer and Tiao (1982), which we hereafter cite as HT: the ARIMA(0, 0, 1)(0, 1, 1)_s model and the ARIMA(0, 1, 1)(0, 1, 1)_s (airline) model. Values considered for the seasonal period s are 2 (biannual), 4 (quarterly), and 12 (monthly). Section 4 discusses some additional related results for canonical ARIMA model-based adjustment, while Section 5 briefly considers special cases for structural component models. Technical details of the derivations for Sections 2 and 3 are reserved to two Appendices.

2 Results for the ARIMA(0, 0, 1)(0, 1, 1)_s model

The ARIMA(0, 0, 1)(0, 1, 1)_s model is

\[(1 − B^s)y_t = (1 − \theta_1 B)(1 − \theta_2 B^s)a_t.\]  

(11)

The nonseasonal and seasonal MA parameters θ_1 and θ_2 are both restricted to lie in the interval (−1, 1), though for seasonal adjustment interest focuses on the case of θ_2 ≥ 0, for which the existence of the canonical decomposition is assured (HT, p. 68). Without loss of generality for the derivations and results presented here, we assume that Var(a_t) = 1.

HT’s canonical decomposition starts with a partial fractions decomposition of the ACGF for y_t.
For the model (11), HT (p. 68) observe that the seasonal part of this partial fractions decomposition can be expressed as $Q_s^*(B)/U(B)U(F)$, where

$$Q_s^*(B) = \frac{(1 - \theta_2)^2(1 - \theta_1 B)(1 - \theta_1 F)}{(1 - B)(1 - F)} \left\{ 1 - \frac{1}{s^2} U(B)U(F) \right\}.$$  (12)

Appendix A observes that $1 - 1/s^2 U(B)U(F)$ contains $(1 - B)(1 - F)$, and so can be expressed as $(1 - B)(1 - F)\alpha_s(B)$, where $\alpha_s(B)$ is a symmetric polynomial in $B$ and $F$. Appendix A also gives $\alpha_s(B)$ for the cases of $s = 2, 4, 12$. Cancelling the $(1 - B)(1 - F)$ factors in the numerator and denominator, $Q_s^*(B)$ simplifies to $(1 - \theta_2)^2(1 - \theta_1 B)(1 - \theta_1 F)\alpha_s(B)$. The spectrum of the canonical seasonal is then $(2\pi)^{-1}$ times $f_s(\lambda) = Q_s^*(e^{i\lambda})/|U(e^{i\lambda})|^2 - \epsilon_s$, where

$$\epsilon_s \equiv \min_{\lambda \in [0, \pi]} \frac{Q_s^*(e^{i\lambda})}{|U(e^{i\lambda})|^2} = \min_{\lambda \in [0, \pi]} \frac{(1 - \theta_2)^2[(1 + \theta_1^2) - 2\theta_1 \cos(\lambda)]\alpha_s(e^{i\lambda})}{|U(e^{i\lambda})|^2}.$$  

The value $\epsilon_s$ becomes part of the canonical irregular variance. If the minimum value $\epsilon_s$ occurs at $\lambda = 0$, then the resulting canonical seasonal spectrum $(2\pi)^{-1} f_s(\lambda)$ will be zero at $\lambda = 0$, and the pseudo-ACGF of $S_t$, which is $\gamma_u(B)/U(B)U(F)$, must include a $1 - B$ factor in $\gamma_u(B)$ (so that $\gamma_u(e^{i0}) = \gamma_u(1) = 0$). By symmetry of $\gamma_u(B)$, it must then also include a $1 - F$ factor, and so in such cases the canonical seasonal filter $\omega_S(B)$ given by (7) will include an extra $(1 - B)(1 - F)$ in its numerator. In these cases the canonical $\omega_S(B)$ for the $(0, 0, 1)(0, 1, 1)_s$ model includes in total $(1 - B)^2(1 - F)^2$. Then $\omega_S(B)$ will annihilate, and $\omega_N(B)$ will reproduce, cubic polynomials in $t$, not just linear polynomials (the standard result for this model which has $d = 1$).

For given values of the nonseasonal MA parameter $\theta_1$, the value of $\lambda$ that minimizes $f_s(\lambda)$ was determined through inspection by computing $f_s(\lambda)$ over a detailed grid of $\lambda$ values (from 0 to $\pi$ in increments of .01) and picking off the minimizing value of $\lambda$. The grid used avoids the exact seasonal frequencies as these are zeros of $U(e^{i\lambda})$, so the function is undefined there. Examining the results for a detailed set of $\theta_1$ values revealed those values of $\theta_1$ for which the minimum of $f_s(\lambda)$ occurs at $\lambda = 0$. These are the values of $\theta_1$ for which $\omega_S(B)$ from the $(0, 0, 1)(0, 1, 1)_s$ model contains $(1 - B)^2(1 - F)^2$ and not just $(1 - B)(1 - F)$. Table 1 below gives the results. Note that,
for $s = 2$, $\omega_S(B)$ contains $(1 - B)^2(1 - F)^2$ for any value of $\theta_1$, while for $s = 4$ and $s = 12$, $\omega_S(B)$ contains $(1 - B)^2(1 - F)^2$ only for limited intervals of $\theta_1$. Note also that the value of $\theta_2$ does not affect the results.

Table 1. Range of values of $\theta_1$ for which the canonical seasonal filter $\omega_S(B)$ from (7) for the ARIMA$(0, 0, 1)(0, 1, 1)_s$ model (11) includes $(1 - B)^2(1 - F)^2$, not just $(1 - B)(1 - F)$.

<table>
<thead>
<tr>
<th>seasonal period $s$</th>
<th>2</th>
<th>4</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>range of values of $\theta_1$</td>
<td>all $\theta_1 \in (-1, 1)$</td>
<td>$-0.35 &lt; \theta_1 &lt; 1$</td>
<td>$-0.28 &lt; \theta_1 &lt; 1$</td>
</tr>
</tbody>
</table>

3 Results for the ARIMA$(0, 1, 1)(0, 1, 1)_s$ (airline) model

The ARIMA$(0, 1, 1)(0, 1, 1)_s$ (airline) model is (Box and Jenkins 1970, Section 9.2)

$$ (1 - B)(1 - B^s)y_t = (1 - \theta_1 B)(1 - \theta_2 B^s)a_t. $$

As with the $(0, 0, 1)(0, 1, 1)_s$ model, the nonseasonal and seasonal MA parameters $\theta_1$ and $\theta_2$ are restricted to lie in the interval $(-1, 1)$, though again interest focuses on the case of $\theta_2 \geq 0$, for which existence of the canonical decomposition is assured. We again assume without loss of generality that $\text{Var}(a_t) = 1$.

HT (p. 67) observe that, for $y_t$ following the model (13) with $\theta_2 \geq 0$, the seasonal part of the
partial fractions decomposition of $\gamma_y(B)$ can be expressed as $Q_s^*(B)/U(B)U(F)$, where now

$$Q_s^*(B) = \frac{(1 - \theta_2)^2}{(1 - B)^2(1 - F)^2} \times$$

$$\left\{ \frac{(1 - \theta_1)^2}{4}(1 + B)(1 + F) \left[ 1 - \frac{1}{s^2}U(B)U(F) - \frac{s^2 - 1}{12s^2}(1 - B^s)(1 - F^s) \right] \right\} \quad (14)$$

$$+ \frac{(1 + \theta_1)^2}{4}(1 - B)(1 - F) \left[ 1 - \frac{1}{4s^2}U(B)U(F)(1 + B)(1 + F) \right] \}.$$

Appendix B simplifies the expression in braces in (14), showing that both of its terms contain $(1 - B)^2(1 - F)^2$, so that after cancellation with the $(1 - B)^2(1 - F)^2$ of the denominator, $Q_s^*(B)$ simplifies to

$$Q_s^*(B) = (1 - \theta_2)^2 \left\{ \frac{(1 - \theta_1)^2}{4}(1 + B)(1 + F)m_{s1}(B) + \frac{(1 + \theta_1)^2}{4}m_{s2}(B) \right\}$$

where $m_{s1}(B)$ and $m_{s2}(B)$ are symmetric polynomials given in Appendix B. As in the previous section, the spectrum of the canonical seasonal is then $(2\pi)^{-1}$ times $f_s(\lambda) = Q_s^*(e^{i\lambda})/|U(e^{i\lambda})|^2 - \epsilon_s$, where now

$$\epsilon_s \equiv \min_{\lambda \in [0, \pi]} \frac{Q_s^*(e^{i\lambda})}{|U(e^{i\lambda})|^2}$$

$$= \min_{\lambda \in [0, \pi]} \frac{(1 - \theta_2)^2}{4} \left\{ \frac{(1 - \theta_1)^2}{4}2[1 + \cos(\lambda)]m_{s1}(e^{i\lambda}) + \frac{(1 + \theta_1)^2}{4}m_{s2}(e^{i\lambda}) \right\}.$$

For $s = 2, 4,$ and $12$, and for a detailed set of values of $\theta_1$, the minima $\epsilon_s$ were again determined by inspection, noting cases when the minimum occurred at $\lambda = 0$. As with the $(0, 0, 1)(0, 1, 1)_s$ model, the value of $\theta_2$ does not affect these results. The results revealed the values of $\theta_1$ for which $\gamma_u(B)$ contains $(1 - B)(1 - F)$, implying that $\omega_S(B)$ contains $(1 - B)^3(1 - F)^3$ and not just $(1 - B)^2(1 - F)^2$. Table 2 below gives the results. Analogously to Table 1 of Section 2, we see that, for $s = 2$, $\omega_S(B)$ contains $(1 - B)^3(1 - F)^3$ for any value of $\theta_1$, while for $s = 4$ and $s = 12$, this occurs only for limited intervals of $\theta_1$. However, the limited intervals for $s = 4$ and $s = 12$
given in Table 2 are much smaller than the corresponding intervals given in Table 1.

Table 2. Range of values of $\theta_1$ for which the canonical seasonal filter $\omega_S(B)$ from (7) for the ARIMA$(0, 1, 1)(0, 1, 1)_s$ (airline) model (13) includes $(1 - B)^3(1 - F)^3$, not just $(1 - B)^2(1 - F)^2$.

<table>
<thead>
<tr>
<th>seasonal period $s$</th>
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<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>range of values of $\theta_1$</td>
<td>all $\theta_1 \in (-1, 1)$</td>
<td>$11 &lt; \theta_1 &lt; 1$</td>
<td>$.58 &lt; \theta_1 &lt; 1$</td>
</tr>
</tbody>
</table>

4 Additional results for canonical ARIMA model-based seasonal adjustment

For any seasonal ARIMA model for which the canonical decomposition exists one can obviously check for the presence of additional unit root factors in the various filters by examining the component models from the canonical decomposition. The computations can be done with the original SEATS program (Gomez and Maravall 1997) or the X-13-ARIMA-SEATS program (Monsell 2007), either of which will provide output tables giving the roots of the AR and MA polynomials of the component models. This approach was used to check some results of the previous two sections.

This approach was also applied to the $(1, 1, 0)(0, 1, 1)_{12}$ model $(1 - \phi B)(1 - B)(1 - B^{12})y_t = (1 - \theta B^{12}) a_t$, to check for the presence of an extra $(1 - B)(1 - F)$ factor in the symmetric seasonal filter. For this model the results turn out to depend on the value of the seasonal moving average parameter $\theta$, as well as on the value of $\phi$. For $\theta = .7$, the extra $(1 - B)(1 - F)$ factor was found to be present for $\phi < -.6$, while for $\theta = .8$, it was found for $\phi \leq -.5$. This serves to illustrate that the extra $(1 - B)(1 - F)$ factor in the seasonal filter can indeed occur for models other than the two considered explicitly in Sections 2 and 3.

As noted earlier, for models of the form of (2) with $\sigma^2_\varepsilon > 0$, extra unit root factors are not present in the symmetric canonical irregular filter, and so the symmetric canonical trend filter will
reproduce only polynomials up to degree $2d - 1$, not degree $2d + 1$. For models with $d = 2$ and when $\omega_S(B)$ does contain the extra $(1 - B)(1 - F)$, $\omega_S(B)$ then contains $(1 - B)^3(1 - F)^3$ while $\omega_T(B)$ contains only $(1 - B)^2(1 - F)^2$, so $\omega_T(B)$ reproduces quintic polynomials in $t$ while $\omega_T(B)$ reproduces only cubic polynomials. This matches the results for X-11 symmetric filters reported in Bell (2012, p. 449).

The quotation in Section 1 noted that HT considered a model for which the canonical trend model had a $1 + B$ factor in its MA polynomial. This implies that $\gamma_v(B)$ contains $(1 + B)(1 + F)$, so that $\omega_T(B)$ given by (9) has this extra $(1 + B)(1 + F)$. The quotation refers to HT’s treatment of the $(0, 0, 0)(0, 1, 1)_s$ model, which is the $(0, 0, 1)(0, 1, 1)_s$ model with $\theta_1 = 0$. In fact, HT’s derivations for the $(0, 0, 1)(0, 1, 1)_s$ and the $(0, 1, 1)(0, 1, 1)_s$ models (the latter with $\theta_2 \geq 0$) show that the canonical trend spectrum is minimized at $\lambda = \pi$. Thus, for both these models $\gamma_v(B)$ contains $(1 + B)(1 + F)$, so that $\omega_T(B)$, which always contains $U(B)U(F)$, has this extra $(1 + B)(1 + F)$, and so includes $(1 + B)^2(1 + F)^2$.

Extra $1 - B$ factors will not be present in asymmetric seasonal filters because application of such filters is equivalent to application of the corresponding symmetric seasonal filter $\omega_S(B)$ after forecast and backcast extension of the time series. Since the forecast and backcast extension will reproduce polynomials only up to degree $d - 1$, this becomes the limiting factor in the degree of polynomials reproduced by the asymmetric seasonal adjustment and trend filters (Bell 2012, p. 447). The same argument applies to seasonal unit root factors contained in the asymmetric seasonal adjustment, trend, and irregular filters. For example, though we just noted that $\omega_T(B)$ from the models examined by HT will include $(1 + B)^2(1 + F)^2$ instead of just the expected $(1 + B)(1 + F)^2$, the asymmetric trend filters will include just the single $1 + B$ factor.

The symmetric finite filters (the filters applied at $t = m + 1$ for a time series of length $2m + 1$) provide some further exceptions to the results for model-based adjustment from both canonical ARIMA and from structural component models. For the case of $d = 1$, all the finite seasonal and irregular filters will include $1 - B$, and so will annihilate linear polynomials in $t$, which are
then reproduced by the corresponding finite seasonal adjustment and trend filters (Bell 2012, Table 1). However, the finite symmetric seasonal and irregular filters must, by symmetry, then include \((1 - B)(1 - F)\), so they will annihilate linear polynomials in \(t\), which are then reproduced by the symmetric finite seasonal adjustment and trend filters. The symmetry argument extends to odd values of \(d > 1\). Thus, for \(d = 3\), the symmetric finite seasonal and irregular filters cannot include just \((1 - B)^3\), so they must include \((1 - B)^2(1 - F)^2\). Hence, they will annihilate cubics, not just quadratics. Values of \(d \geq 3\) are seldom used in time series models, however. Finally, since all the finite trend filters include \(U(B)\) which includes the factor \(1 + B\), the symmetric finite trend filters must include \((1 + B)(1 + F)\) (Findley and Martin 2006, p. 29).

5 Special cases for structural component models

Special case results for the structural models proposed by the references cited in Section 1 differ from the special case results presented for canonical ARIMA seasonal adjustment. For the structural models a zero in the spectrum of a component will, in most cases, arise only if model fitting estimates zero for the variance of the component’s stationary part — \(u_t, v_t, \text{or } I_t\) in (2). If that happens, the component becomes deterministic, not stochastic. If \(\sigma^2_I = 0\), then \(I_t = 0\), so it can be dropped from the model, and \(N_t = T_t\). Assuming no other components have zero variances, the formulas (7)–(9) still apply (although (8) and (9) are now the same), and the results of Bell (2012) still apply to signal extraction estimation of \(S_t\) and \(N_t = T_t\).

If \(\text{var}(v_t)\) is estimated to be zero, the fitted model then has \((1 - B)^dT_t = 0\), implying that \(T_t\) is a polynomial in \(t\) of degree \(d - 1\). We cannot leave the component model as \((1 - B)^dT_t = v_t\) with \(\text{var}(v_t) = 0\) and apply (9) since, from (6), setting \(\gamma_v(B) = 0\) will produce a factor of \((1 - B)^d(1 - F)^d\) in \(\gamma_w(B)\), violating an assumption that underlies the symmetric signal extraction formulas (7)–(10), as well as the corresponding asymmetric infinite filter formulas. Instead we replace the stochastic component \(T_t\) in the model by a polynomial regression function \(\beta_0 + \beta_1 t + \cdots + \beta_{d-1} t^{d-1}\).

The fitted value of this function provides \(\hat{T}_t\), and the signal extraction estimate of \(N_t\) is then
\[ \hat{T}_t + \omega_I(B)[y_t - \hat{T}_t] \] (assuming \( \hat{\sigma}_I^2 > 0 \)). If this form of signal extraction estimation (including regression estimation of the \( \beta_j \)'s) is applied to a time series \( y_t \) that is exactly a polynomial in \( t \) of degree \( d - 1 \) or less, the polynomial will be reproduced in \( \hat{T}_t \), and thus also in \( \hat{N}_t \). This contrasts with the symmetric infinite filter estimates for seasonal adjustment and trend estimation that apply with \( \text{var}(v_t) > 0 \), which reproduce polynomials of degree \( 2d - 1 \). For related discussion on treatment of trend constants, see Bell (2010, pp. 5-6), including the proof given of Theorem 2.

Having \( \text{var}(v_t) = 0 \) is acceptable for finite sample signal extraction, but will produce the same results as modeling \( T_t \) as a \( d - 1 \) degree polynomial regression function. Analogous results to those just described hold if \( u_t \) is estimated to have zero variance so \( S_t \) becomes fixed seasonal effects. See Harvey (1981) and Bell (1987) for discussion related to these two points.

Special case results are more involved for the local linear trend model of Harvey (1989, p. 37), which is

\[ (1 - B)T_t = \beta_t + \varepsilon_{1t} \quad \text{where} \quad (1 - B)\beta_t = \varepsilon_{2t} \]

with \( \varepsilon_{1t} \) and \( \varepsilon_{2t} \) independent white noise series with variances \( \sigma_{\varepsilon_1}^2 \) and \( \sigma_{\varepsilon_2}^2 \). This model can be rewritten as ARIMA(0,2,1): \( (1 - B)^2T_t = (1 - \eta B)c_t \), with \( \eta \in [0,1] \). If \( \sigma_{\varepsilon_2}^2 > 0 \) then \( \eta < 1 \) and the usual results of Bell (2012) apply: the symmetric infinite filters \( \omega_S(B) \) and \( \omega_I(B) \) contain \( (1 - B)^2(1 - F)^2 \), and \( \omega_N(B) \) and \( \omega_T(B) \) reproduce cubics. If both \( \sigma_{\varepsilon_1}^2 \) and \( \sigma_{\varepsilon_2}^2 \) equal 0, then \( (1 - B)^2T_t = 0 \) and, from the previous discussion, \( T_t = \beta_0 + \beta_1t \) and estimation of \( N_t \) and \( T_t \) reproduces only linear functions of \( t \). If \( \sigma_{\varepsilon_1}^2 > 0 \) but \( \sigma_{\varepsilon_2}^2 = 0 \), then \( \beta_t \) becomes a fixed trend constant \( \beta \), and the model becomes a random walk with a trend constant, for which estimation of \( N_t \) and \( T_t \) again reproduces just linear polynomials. To summarize, if \( \sigma_{\varepsilon_2}^2 > 0 \), then \( \omega_N(B) \) and \( \omega_T(B) \) in (8) and (9) reproduce cubics, while if \( \sigma_{\varepsilon_2}^2 = 0 \), then asymmetric and finite sample signal extraction estimation of \( N_t \) and \( T_t \) reproduce only linear functions of \( t \). Note that estimating \( \sigma_{\varepsilon_2}^2 = 0 \) but \( \sigma_{\varepsilon_1}^2 > 0 \), which is equivalent to estimating \( \eta = 1 \) in the ARIMA(0,2,1) formulation, occurs frequently in practice (Bell and Pugh 1990, Shephard 1993).
Appendix A: Derivation details for the ARIMA \((0, 0, 1)(0, 1, 1)_s\) model

We consider (12):

\[
Q_s^*(B) = \frac{(1 - \theta_2)^2(1 - \theta_1 B)(1 - \theta_1 F)}{(1 - B)(1 - F)} \left\{ 1 - \frac{1}{s^2}U(B)U(F) \right\}.
\]

Applying \(U(B)\) or \(U(F)\) to the constant \(s\), so that applying \(1 - 1/s^2U(B)U(F)\) to \(1\) yields 0. This shows that \(1 - 1/s^2U(B)U(F)\) contains a factor \((1 - B)\). Since

\[
1 - \frac{1}{s^2}U(B)U(F) = \frac{1}{s^2} \left[ s(s - 1) - (s - 1)(B + F) - (s - 2)(B^2 + F^2) - \cdots - 2(B^{s-2} + F^{s-2}) - (B^{s-1} + F^{s-1}) \right]
\]

has symmetric coefficients, it must also contain \((1 - F)\), and so can be expressed as

\((1 - B)(1 - F)\alpha_s(B)\), where the polynomial \(\alpha_s(B)\), which is of degree \(s - 2\) in \(B\) and \(F\), also has symmetric coefficients. Cancelling the \((1 - B)(1 - F)\) factors in the numerator and denominator of \(Q_s^*(B)\) then simplifies it to \((1 - \theta_2)^2(1 - \theta_1 B)(1 - \theta_1 F)\alpha_s(B)\).

The coefficients of \(\alpha_s(B)\) can be obtained using the following Lemma on division of polynomials in \(B\) by \(1 - B\) and \(1 - F\).

**Lemma:** Let \(a(B) = a_0 + a_1 B + \cdots + a_k B^k\) be a polynomial in \(B\) of degree \(k > 0\). Then

(i) \[ \frac{a(B)}{1 - B} = a_0 + (a_0 + a_1)B + \cdots + (a_0 + \cdots + a_{k-1})B^{k-1} + \frac{(a_{k-1} + \cdots + a_k)B^k}{1 - B}, \]

(ii) \[ \frac{a(B)}{1 - F} = a_k B^k + (a_k + a_{k-1})B^{k-1} + \cdots + (a_k + \cdots + a_1)B + \frac{(a_0 + \cdots + a_k)}{1 - F}. \]

If \(a_0 + \cdots + a_k = 0\), then \(a(B)\) contains \(1 - B\) (equivalently, contains \(1 - F\)) as a factor.

**Proof:** Results (i) and (ii) are easily verified by writing their right-hand side expressions as single fractions and simplifying. The last statement follows since the remainder terms in (i) and (ii) are zero if \(a_0 + \cdots + a_k = 0\).

Note from the Lemma that the coefficients of the \(k - 1\) degree polynomial that results from dividing \(a(B)\) by \(1 - B\) can be obtained by cumulatively summing the coefficients of \(a(B)\) or, for
division by \( 1 - F \), by cumulatively summing the coefficients of \( a(B) \) in reverse order. Also, note that the Lemma can be applied to a polynomial in \( B \) and \( F \). Thus, if \( a(B) = a_{-j}F^j + \cdots + a_{-1}F + a_0 + a_1B + \cdots + a_kB^k \), we pre-multiply \( a(B) \) by \( B^j \), where \( j \) is the highest power of \( F \) in \( a(B) \), then apply the Lemma, and then multiply the result from the division by \( 1 - B \) or \( 1 - F \) by \( F^j \).

Using the Lemma, we obtained the coefficients of \( \alpha_s(B) \) by cumulatively summing the coefficients of \( 1 - 1/s^2U(B)U(F) \), and then cumulatively summing the resulting coefficients in reverse order. The results of this are as follows for the three values of \( s \) that we consider:

\[
\begin{align*}
    s &= 2: \quad \alpha_2(B) = \frac{1}{4} \\
    s &= 4: \quad \alpha_4(B) = \frac{1}{16} \left[ 10 + 4(B + F) + (B^2 + F^2) \right] \\
    s &= 12: \quad \alpha_{12}(B) = \frac{1}{144} \left[ 286 + 220(B + F) + 165(B^2 + F^2) + 120(B^3 + F^3) \right. \\
    & \quad \quad \quad + 84(B^4 + F^4) + 56(B^5 + F^5) + 35(B^6 + F^6) + 20(B^7 + F^7) \\
    & \quad \quad \quad \left. + 10(B^8 + F^8) + 4(B^9 + F^9) + (B^{10} + F^{10}) \right].
\end{align*}
\]

**Appendix B: Derivation details for the ARIMA \((0, 1, 1)(0, 1, 1)_s\) (airline) model**

For the airline model, we consider (14):

\[
Q_2^*(B) = \frac{(1 - \theta_2)^2}{(1 - B)^2(1 - F)^2} \times \left\{ \frac{(1 - \theta_1)^2}{4}(1 + B)(1 + F) \left[ 1 - \frac{1}{s^2}U(B)U(F) - \frac{s^2 - 1}{12s^2}(1 - B^s)(1 - F^s) \right] \\
+ \frac{(1 + \theta_1)^2}{4}(1 - B)(1 - F) \left[ 1 - \frac{1}{4s^2}U(B)U(F)(1 + B)(1 + F) \right] \right\}.
\]

We know from Appendix A that \( 1 - 1/s^2U(B)U(F) = (1 - B)(1 - F)\alpha_s(B) \). Also, \( (1 - B^s)(1 - F^s) = (1 - B)(1 - F)U(B)U(F) \). The first term in brackets on the right-hand side above is thus \((1 - B)(1 - F)\) times \( \alpha_s(B) - \frac{s^2 - 1}{12s^2}U(B)U(F) \). If, for each of the cases \( s = 2, \ldots, 12 \),
4, and 12, we sum the coefficients of \( \alpha_s(B) - \frac{s^2-1}{12s^2} U(B)U(F) \), and then reverse sum the resulting sequence, we find that the first and last values in this twice summed sequence are both zero. Thus, from the Lemma, \( \alpha_s(B) - \frac{s^2-1}{12s^2} U(B)U(F) = (1 - B)(1 - F)m_{s1}(B) \), where \( m_{s1}(B) \) is the symmetric polynomial whose coefficients are the nonzero terms of the sequence produced by this summing and reverse summing. For the second term in brackets on the right-hand side above, if we sum the coefficients of \( 1 - \frac{1}{4s^2}U(B)U(F)(1+B)(1+F) \), and reverse sum the result, we get zero for the first and last coefficients, so that \( 1 - \frac{1}{4s^2}U(B)U(F)(1+B)(1+F) = (1 - B)(1 - F)m_{s2}(B) \) for the symmetric polynomial \( m_{s2}(B) \) whose coefficients we just produced. The terms in the second and third lines of the expression (14) for \( Q^*_s(B) \) thus both contain \((1 - B)^2(1 - F)^2\), and cancelling this with the \((1 - B)^2(1 - F)^2\) in the denominator shows that

\[
Q^*_s(B) = (1 - \theta_2)^2 \left\{ \frac{(1 - \theta_2)^2}{4} (1 + B)(1 + F)m_{s1}(B) + \frac{(1 + \theta_1)^2}{4} m_{s2}(B) \right\}.
\]

The polynomials \( m_{s1}(B) \) and \( m_{s2}(B) \) for the cases of \( s = 2, 4, \) and 12 are given below.

\( s = 2: \quad m_{2,1}(B) = \frac{1}{4} \quad \text{and} \quad m_{2,2}(B) = \frac{1}{16}(6 + B + F) \)

\( s = 4: \quad m_{4,1}(B) = \frac{3}{16} \left[ 26 + 16(B + F) + 5(B^2 + F^2) \right] \)

\[
m_{4,2}(B) = \frac{1}{64} \left[ 44 + 19(B + F) + 6(B^2 + F^2) + (B^3 + F^3) \right]
\]

\( s = 12: \quad m_{12,1}(B) = \frac{1}{1,728} \left[ 16, 874 + 16, 016(B + F) + 14, 091(B^2 + F^2) \right.
\]

\[
+ 11, 616(B^3 + F^3) + 8, 988(B^4 + F^4) + 6, 496(B^5 + F^5)
\]

\[
+ 4, 333(B^6 + F^6) + 2, 608(B^7 + F^7) + 1, 358(B^8 + F^8) \n\]
\[
m_{12,2}(B) = \frac{1}{576} [1,156 + 891(B + F) + 670(B^2 + F^2) + 489(B^3 + F^3) \\
+ 344(B^4 + F^4) + 231(B^5 + F^5) + 146(B^6 + F^6) \\
+ 85(B^7 + F^7) + 44(B^8 + F^8) + 19(B^9 + F^9) \\
+ 6(B^{10} + F^{10}) + (B^{11} + F^{11})].
\]

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**References**


